Differentiating this result with respect to x and y, we obtain

$$\frac{\partial}{\partial x}|\sin z|^2 = 2\sin x\cos x\cosh^2 y - 2\sin x\cos x\sinh^2 y = 2\sin x\cos x = \sin 2x$$

and

$$\frac{\partial}{\partial y}|\sin z|^2 = 2\sin^2 x\sinh y\cosh y + 2\cos^2 x\sinh y\cosh y = 2\sinh y\cosh y = \sinh 2y.$$

The maxima and minima of $|\sin z|^2$, and hence of $|\sin z|$, will occur in D if each of these derivatives vanishes simultaneously at a point or points inside D. The function $\sin 2x$ only vanishes in D on the line $x = \pi/2$, but $\sinh 2y \neq 0$ for $0 < y < 1$, so $|\sin z|^2$, and hence $|\sin z|$, has neither maxima nor minima in D. Thus, the extrema of $|\sin z|$ must occur on the straight line boundaries of D. On the boundary $x = 0$ of D, $|\sin z|$ has a minimum of 0 at $(0, 0)$ and a maximum of $\sinh 1$ at $(0, 1)$. On the boundary $x = \pi$ of D, $|\sin z|$ has a minimum of 0 at $(\pi, 0)$ and a maximum of $\sinh 1$ at $(\pi, 1)$. On the boundary $y = 0$ of D, $|\sin z|$ has two minima of 0 at $(0, 0)$ and $(0, \pi)$ and a maximum of 1 at $(\pi/2, 0)$, while on the boundary $y = 1$ of D, $|\sin z|$ has two minima equal to $\sinh 1$ at $(0, 1)$ and $(\pi, 1)$, and a maximum of $(1 + \sinh^2 1)^{1/2}$ at $(\pi/2, 1)$. This shows that the smallest value of $|\sin z|$ on the boundary of D is 0, and the largest value is $(1 + \sinh^2 1)^{1/2}$.

The results of Theorem 14.17 are confirmed, so inside the rectangle D it follows that

$$0 < |\sin z| < (1 + \sinh^2 1)^{1/2}, \quad \text{for all } z \text{ inside } D.$$

■

We now use Theorem 14.17 to prove a corresponding result for harmonic functions that has important consequences in the study of boundary value problems for Laplace's equation.

THEOREM 14.18

an extremum principle for a harmonic function u

The maximum/minimum principle for harmonic functions The maximum and minimum values of a nonconstant function u that is harmonic in a bounded simply connected domain and continuous on its boundary must occur on the boundary.

Proof Let u be a harmonic function satisfying the conditions of the theorem, and form the analytic function $f(z) = u + iv$, where v is the harmonic conjugate of u. Then,

$$|\exp\{f(z)\}| = |e^{u+iv}| = |e^u||e^{iv}| = e^u.$$

As e^u is a monotonic increasing function of u, this result shows that the maxima of e^u, and hence of u and those of $|\exp f(z)|$, coincide. Using this result in Theorem 14.16 shows that the maxima of u must occur on the boundary. The fact that the minima of u also occur on the boundary follows if we notice that the minima of u correspond to the maxima of the harmonic function $-u$, so the proof is complete. ■

Theorem 14.18 also applies to nonsimply connected bounded domains. In such a domain D, the maximum value of u is taken to be the largest of the maxima on all the internal boundaries and the external boundary of D, and the minimum value is taken to be the smallest of the minima on all the internal boundaries and the external boundary of D.

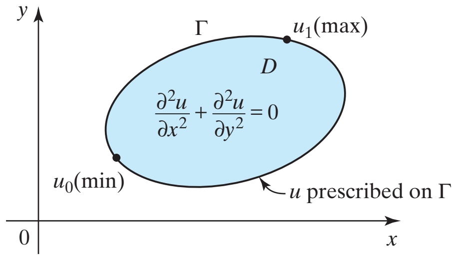


FIGURE 14.22 A two-dimensional boundary value problem for Laplace's equation.

To see how the theorem provides qualitative information about solutions of Laplace's equation $u_{xx} + u_{yy} = 0$, consider the bounded two-dimensional domain D with boundary Γ shown in Fig. 14.22 on which u assumes prescribed continuous values, and let the smallest of the values of u on Γ be u_0 and the largest be u_1. Then, for all points (x, y) in D we have

$$u_0 < u(x, y) < u_1.$$

Problems of this type are called two-dimensional **boundary value problems** for Laplace's equation. They occur, for example, when a two-dimensional steady-state temperature distribution is to be determined within a uniform heat-conducting medium on the boundary of which the temperature takes prescribed values, because the steady state temperature as a function of position in the medium is a solution of Laplace's equation.

EXAMPLE 14.17

Use Theorem 14.18 to place bounds on the function $u(x, y) = (1 + 2\sinh^2 x)\sin 2y$ in the domain D determined by $0 \le x \le 1$ and $0 \le y \le \pi$.

Solution Routine differentiation establishes that $u_{xx} + u_{yy} = 0$, so $u(x, y)$ is harmonic. As the domain D is bounded and $u(x, y)$ is harmonic, Theorem 14.18 applies and asserts that the smallest and largest values of $u(x, y)$ must occur on the boundary of D. Examination of the behavior of $u(x, y)$ on the straight line boundaries of D shows that the smallest value of $u(x, y)$ is $-1 - 2\sinh^2 1$ at $(1, 3\pi/4)$, and the largest value is $1 + 2\sinh^2 1$ at $(1, \pi/4)$, so

$$-1 - 2\sinh^2 1 < u(x, y) < 1 + 2\sinh^2 1 \text{ at all points inside } D.$$

The next example illustrates how the maximum/minimum principle may be used to place bounds on the two-dimensional temperature distribution inside a long uniform hexagonal rod of metal when an arbitrary temperature distribution is prescribed around its hexagonal faces. The bounds on the temperature distribution inside the metal can, for example, be used to estimate the thermal stress produced in the rod due to the uneven heating of its faces.

EXAMPLE 14.18

Consider the cross-section of a long hexagonal rod of metal, shown in Fig. 14.23a, where the inscribed circle that is tangent to the faces has radius $a\sqrt{3}/2$, and the circumscribed circle that passes through the vertices has radius a. Draw a ray from the origin to a point on the circumscribed circle, and let $T = f(\theta)$ be the temperature

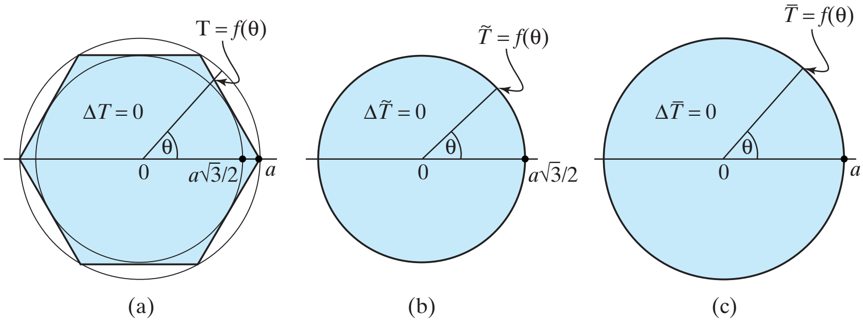


FIGURE 14.23 The hexagonal cross-section and two related cross-sections with radii $a\sqrt{3}/2$ and a.

that is imposed on the hexagonal face where the ray intersects the face. Then the function $f(\theta)$ is periodic with period 2π.

We now anticipate the result of Chapter 17 (proved in Chapter 18) that the steady state temperature distribution in a uniform heat conducting medium satisfies the Laplace equation. As the problem is two-dimensional, it follows that the temperature distribution inside the hexagonal cross-section must satisfy the two-dimensional Laplace equation $\Delta T = 0$.

Our approach will be to consider two related but far simpler problems than the problem in the hexagonal cross-section. One will be for the Laplace equation for a temperature $\tilde{T}(r,\theta)$ inside the inscribed circle, and the other for a temperature $\bar{T}(r,\theta)$ inside the circumscribed circle, when both problems satisfy the same temperature distribution at an angle θ on the perimeter of their respective circles as the temperature on the plane face at the same angle. We start by considering the problem in cylindrical polar coordinates $\Delta\tilde{T}(r,\theta) = 0$ in the disc of radius $a\sqrt{3}/2$ shown in Fig. 14.23b that is required to satisfy the temperature $\tilde{T}(a\sqrt{3}/2,\theta) = f(\theta)$ on the perimeter of the circle. Then, as the temperatures on the hexagonal faces have been transferred *inward* to corresponding points on the inscribed circle, it follows directly from the maximum/minimum principle that inside and on the inscribed circle we must have $T(r,\theta) \le \tilde{T}(r,\theta)$. Thus, $\tilde{T}(r,\theta)$ provides an upper bound for the temperature in any cross-section of the hexagonal rod at points that lie inside the circle of radius $a\sqrt{3}/2$.

Next, we consider the corresponding problem shown in Fig. 14.23c, where this time the solution $\bar{T}(a,\theta)$ of the Laplace equation inside the circumscribed circle is required to satisfy the temperature $\bar{T}(a,\theta) = f(\theta)$ on the perimeter of the circle. Here the temperatures on the hexagonal faces have been transferred *outward* to corresponding points on the circumscribed circle, so this time by the maximum/minimum principle it follows that $\bar{T}(r,\theta) \le T(r,\theta)$. Thus, $\bar{T}(r,\theta)$ provides a lower bound for the temperature $\bar{T}(r,\theta)$ at all points inside the hexagonal cross-section.

Consequently, we have established the following results:

(i) $\bar{T}(r,\theta) \le T(r,\theta)$ at all points inside the hexagonal cross-section

(ii) $T(r,\theta) \le \tilde{T}(r,\theta)$ at all points inside the hexagonal cross-section that belong to the inscribed circle

To make further progress we appeal to the Poisson integral formula for a circle that forms the result of Exercise 3 in Exercise Section 14.4. This asserts that if $u(r,\theta)$

is harmonic in a circle of radius R centered on the origin, and on the perimeter of the circle $u(R, \theta) = f(\theta)$, then

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2) f(\psi)}{R^2 - 2rR\cos(\theta-\psi) + r^2} d\psi.$$

The bound $\tilde{T}(r, \theta)$ follows directly from this result by setting $R = a\sqrt{3}/2$ and $u(r,\theta) = \tilde{T}(r,\theta)$, while the bound for $\bar{T}(r,\theta)$ follows by setting $R = a$ and $u(r,\theta) = \bar{T}(r,\theta)$. Clearly, this approach works for any cross-section shape, though the bounds will be sharper when the radii of the inscribed and circumscribed circles are close together. ■

The performance of an engineering system often depends on the location of the zeros of a function that may not necessarily be a polynomial. To obtain a system with satisfactory properties, the zeros are often required to lie in a particular part of the z-plane. This occurs, for example, when working with control systems governed by a system of differential equations, because the system will only be stable if the zeros of a characteristic equation all lie to the left of the imaginary axis, and so have negative real parts. However, to avoid an undesirably slow decay of any disturbances to such a system, it is usually also necessary to require that each zero have a real part that is less than some prescribed negative number, so in such cases all zeros must lie to the left of a line $z = -c$ with $c > 0$. Consequently, when such a system has parameters that can be adjusted to optimize performance, unless the zeros can be found explicitly, it is necessary to devise a practical test that determines how many zeros lie inside a given region contained within a closed curve Γ.

A powerful test of this type can be derived from the following result we will call the **restricted argument principle**, as it is a special case of what in complex analysis is known as the **argument principle**. Although this more general theorem is not difficult to establish, its proof would be out of place here and will be omitted, as it can be found in any of the references quoted at the end of this chapter.

THEOREM 14.19

The restricted argument principle Let $f(z)$ be analytic and have a finite number of zeros and no poles in a bounded simply connected domain D with boundary Γ. Then, provided $f(z) \neq 0$ on Γ,

$$\frac{1}{2\pi}\Delta_\Gamma \arg f(z) = N,$$

where $\Delta_\Gamma \arg f(z)$ denotes the change in the argument of $f(z)$ when the contour Γ is traversed once in the positive (counterclockwise) sense, and N is the number of zeros in D with their multiplicity counted. ■

The geometrical implication of this theorem is as follows. Let Γ' be the image of Γ under the mapping $w = f(z)$. Then, when a point z makes one traverse of the contour Γ in the z-plane, the number of times its image Γ' encircles the origin in the w-plane is equal to the number of zeros of $f(z)$ inside Γ. To apply this geometrical interpretation of the theorem, the contour Γ in the z-plane must first be parametrized, after which this parametrization must be used in $w = f(z)$ to construct the image Γ' in the w-plane. The number of times Γ' encircles the origin $w = 0$ can then be counted to determine the number of zeros of $f(z)$ inside Γ.

A result that can be derived from the restricted argument principle, which although weaker is both useful and simple to use, is **Rouché's theorem**.

THEOREM 14.20

Rouché's theorem Let D be a simply connected domain bounded by a contour Γ in which the functions $f(z)$ and $g(z)$ are analytic and such that $|f(z)| > |g(z)|$ for all z on Γ. Then $f(z)$ and $f(z)+g(z)$ each have the same number of zeros in D. ■

In effect, the conditions of Rouché's theorem are such that it enables the number of zeros of a simple function $f(z)$ inside Γ to be equated to the number of zeros possessed by the more complicated function $f(z)+g(z)$ that also lie inside Γ.

EXAMPLE 14.19

Use Rouché's theorem to find the number of zeros of the polynomial $P(z) = z^4 - 8z + 10$ that lie (a) in $|z| \le 1$ and (b) in $|z| \le 3$. (c) Confirm results (a) and (b) by using the graphical implication of the restricted argument principle.

Solution

(a) Make the identifications $f(z) = 10$ and $g(z) = z^4 - 8z$. On $|z| = 1$ we have $|f(z)| = 10$ and $|g(z)| = |z^4 - 8z| \le |z| + 8|z| = 9$, so $|f(z)| > |g(z)|$ on $|z| = 1$. Then by Rouché's theorem, as $f(z)$ has no zeros inside $|z| = 1$, it follows that $f(z) + g(z) = P(z)$ has no zeros inside $|z| \le 1$.

(b) Make the identification $f(z) = z^4$ and $g(z) = -8z + 10$. On $|z| = 3$, $|f(z)| = 81$ and $|g(z)| = |-8z + 10| \le 8|z| + 10 = 34$, so $|f(z)| > |g(z)|$ on $|z| = 3$. Then by Rouché's theorem, as $f(z)$ has four zeros inside $|z| = 3$ when their multiplicity is counted, it follows that $f(z) + g(z) = P(z)$ also has four zeros inside $|z| \le 3$.

(c) Parametrize the circle $|z| = 1$ by setting $x = \cos t$, $y = \sin t$ with $0 \le t \le 2\pi$ so the unit circle is traversed once. Then setting $z = \cos t + i \sin t$ in $w = u + iv = P(z)$ and separating out the real and imaginary parts gives

$$u = \cos^4 t - 6\cos^2 t \sin^2 t + \sin^4 t - 8\cos t + 10$$
$$v = 4\cos^3 t \sin t - 4\cos t \sin^3 t - 8\sin t.$$

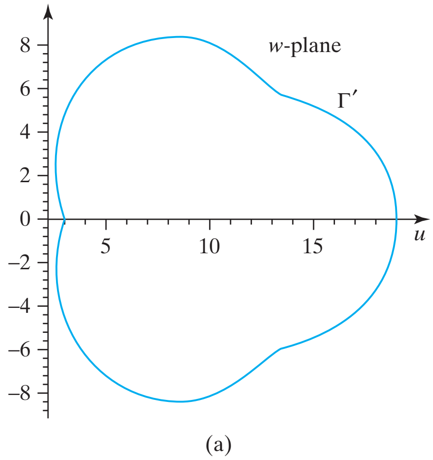
The image Γ' of Γ under the mapping $w = f(z)$ is obtained by plotting this parametric representation of Γ' with $0 \le t \le 2\pi$. This plot is shown in Fig. 14.24a, from which it can be seen that the image Γ' does not encircle the origin in the w-plane, so no zeros of $P(z)$ lie in $|z| = 1$.

Repeating this argument, but this time parametrizing the circle $|z| = 3$ by setting $z = 3(\cos t + i \sin t)$, leads to the results

$$u = 81\cos^4 t - 486\cos^2 t \sin^2 t + 81\sin^4 t - 24\cos t + 10$$
$$v = 324\cos^3 t \sin t - 324\cos t \sin^3 t - 24\sin t.$$

The plot of this image of Γ' is shown in Fig. 14.24b, from which it can be seen that Γ' encircles the origin in the w-plane four times, so $P(z)$ has four zeros inside the circle $|z| = 3$. ■

Alternative accounts and extra information concerning the material in Sections 14.1 to 14.4 can be found in any one of references [6.1] to [6.4] and [6.6] to [6.9].



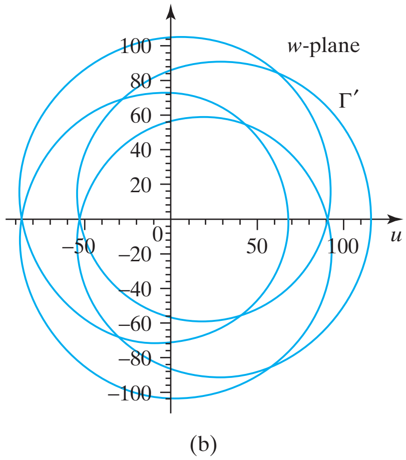


FIGURE 14.24 (a) Γ' does not encircle $w = 0$. (b) Γ' encircles $w = 0$ four times.

Summary

Some general properties of analytic functions were derived, one of which was the fundamental theorem of algebra that asserts every polynomial of degree n has precisely n zeros, though these need not all be distinct. The maximum/minimum modulus theorem for analytic functions was also proved, showing that the maximum and minimum values of the modulus of a nonconstant analytic function defined in a domain D must occur on the boundary of D. A corresponding theorem for harmonic functions was also proved.

EXERCISES 14.4

1.* Let $P_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a complex polynomial, and Γ be a positively oriented circle with its center at the origin. Show that

$$\frac{1}{2\pi i}\sum_{k=0}^{n}\int_{\Gamma}\frac{P_n(z)}{z^{k+1}}dz = \sum_{k=0}^{n} a_k.$$

2.* Let $f(z)$ be analytic inside and on the circle Γ defined by $|z| = R$, and let $z_0 = re^{i\theta}$, with $0 < r < R$, be a point inside the circle. Show that the point $Z = z\bar{z}/\bar{z}_0$ lies outside the circle Γ, so that

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f(z)}{z - Z}dz = 0.$$

By differencing this expression and the expression for $f(z_0)$ determined by the Cauchy integral formula, show that

$$f(re^{i\theta}) = \frac{1}{2\pi i}\int_{\Gamma}\frac{1}{z}\frac{(z\bar{z} - z_0\bar{z}_0)}{(z - z_0)(\bar{z} - \bar{z}_0)}f(z)dz.$$

3.* By setting $z_0 = re^{i\theta}$ and $z = Re^{i\psi}$ in the result of Exercise 2, show that

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2 - r^2)}{R^2 - 2rR\cos(\psi - \theta) + r^2}f(Re^{i\psi})d\psi.$$

Write $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$ in the preceding result and derive the **Poisson integral formula for a disc**,

$$u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2 - r^2)u(R, \psi)}{R^2 - 2rR\cos(\psi - \theta) + r^2}d\psi.$$

This formula determines the value of the harmonic function $u = \text{Re}\{f(z)\}$ at any point (r, θ) inside the disc in terms of the prescribed values of u on the boundary Γ of the disc. The specification of u on the boundary of a domain in which u is harmonic constitutes what is called a **Dirichlet problem** for Laplace's equation. This formula determines, for example, the steady state electrostatic potential in a long cavity with a circular cross-section of radius R, on the walls of which the potential $u(R, \psi) = f(R, \psi)$. As the steady state two-dimensional temperature distribution in a long metal rod of circular cross-section of radius R is also a solution of Laplace's equation, this same formula determines the temperature distribution in the rod when its surface is at a temperature $u(R, \psi) = f(R, \psi)$.

4.* By setting $u(R, \psi) = M$ in the Poisson integral formula for a disc given in Exercise 3, and using the result

$$\int_0^{2\pi} \frac{dt}{1 + a\cos t} = \frac{2\pi}{\sqrt{1 - a^2}} \quad \text{for } a^2 < 1$$

that can be established by the method of Example 14.13, show that when $u(R, \psi) = M$ (constant) on the boundary of the disc, it must follow that $u(r, \theta) \equiv M$ throughout the disc.

5.* Let domain D be the interior of the positively oriented contour Γ comprising the semicircle C_R of radius R in the upper half plane with its center at the origin, and the segment of the real axis from $-R$ to R. If z_0 is an interior point of D, explain why

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - z_0}dz \text{ and } 0 = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - \bar{z}_0}dz.$$

Set $z_0 = x_0 + iy_0$ and difference these results to show that

$$f(z_0) = \frac{y_0}{\pi}\int_{-R}^{R} \frac{f(x)}{|x - z_0|^2}dx + \frac{y_0}{\pi}\int_{C_R} \frac{f(z)}{(z - z_0)(z - \bar{z}_0)}dz.$$

6.* Using the notation of Exercise 5, and writing $z = z_0 + (z - z_0)$ and $z = \bar{z}_0 + (z - \bar{z}_0)$, show that

$$(R - |z_0|)^2 \le |z - z_0| \cdot |z - \bar{z}_0|.$$

Deduce from this that if $|f(z)| \le K$ in the upper half plane, then

$$\left|\frac{y_0}{\pi}\int_{C_R} \frac{f(z)}{(z - z_0)(z - \bar{z}_0)}dz\right| \le \frac{Ky_0R}{(R - |z_0|)^2}.$$

By taking the limit of the result of Exercise 5 as $R \to \infty$ and using the result from this exercise, deduce that

$$f(z_0) = \frac{y_0}{\pi}\int_{-\infty}^{\infty} \frac{f(x)}{(x - x_0)^2 + y_0^2}dx.$$

Then, by setting $f(z) = u(x, y) + iv(x, y)$ and equating the real parts of the equation, show that

$$u(x_0, y_0) = \frac{y_0}{\pi}\int_{-\infty}^{\infty} \frac{u(x, 0)}{(x - x_0)^2 + y_0^2}dx, \quad \text{for } y_0 > 0.$$

This result is the **Poisson integral formula for a half-plane**, and it determines the harmonic function $u(x_0, y_0)$ at points (x_0, y_0) in the upper half-plane in terms of a prescribed function $u(x, 0)$ on the real axis. The function $u(x, 0)$ is called a **Dirichlet boundary condition** for the two-dimensional boundary value problem for Laplace's equation. This formula can be used to determine the steady state temperature distribution $u(x, y)$ in a thermally conducting half-plane when the temperature on the plane bounding surface is $u(x, 0) = T(x)$, with $T(x)$ a given function. A similar interpretation applies when the formula is used to determine the steady state electrostatic potential $u(x, y)$ in a half-space when the potential on the plane bounding surface is $u(x, 0) = T(x)$.

7. Let $P_n(z)$ be the complex polynomial $P_n(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$ with $a_n \ne 0$, and $n \ge 1$. Justify the assertion in the proof of the fundamental theorem of algebra that if $Q_n(z) = 1/P_n(z)$, then $\lim_{|z|\to\infty} |Q_n(z)| = \lim_{|z|\to\infty} 1/|P_n(z)| = 0$.

8. Given that $z = 1 + 2i$ is a root of the polynomial $z^4 + 2z^3 + 10z^2 - 6z + 65 = 0$ with real coefficients, use the deflation method described in the proof of the fundamental theorem of algebra to find the remaining roots.

9. Verify the maximum/minimum principle for the function $f(z) = e^z$ in the domain $-1 \le x \le 1$, $-2 \le y \le 2$, and place bounds on $|e^z|$ inside the given domain.

10. Verify the maximum/minimum principle for the function $f(z) = \cosh z$ in the domain $-1 \le x \le 1, -1 \le y \le 1$, and place bounds on $|\cosh z|$ inside the given domain.

In Exercises 11 through 14 place bounds on the function $u(x, y)$ inside the given domain.

11. $u(x, y) = x + 2x^2 - 2y^2$ in the domain $-1 \le x \le 1$, $-1 \le y \le 1$.

12. $u(x, y) = e^x(y\cos y + x\sin y)$ in the domain $0 \le x \le 1$, $-\pi/2 \le y \le \pi/2$.

13. $u(x, y) = e^x(x\cos y - y\sin y)$ in the domain $0 \le x \le 1$, $-\pi/2 \le y \le \pi/2$.

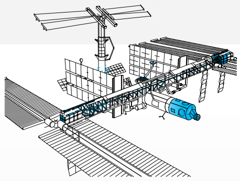
14. $u(x, y) = e^x(\cos^2 y\cosh x - \sin^2 y\sinh x)$ in the domain $0 \le x \le 1, 0 \le y \le \pi/2$.

15. Show by Rouché's theorem that $P(z) = z^4 - 5z + 1$ has one zero in the disc $|z| \le 1$ and three zeros in the annulus $1 \le |z| \le 2$.

16. Use Rouché's theorem to find the number of zeros of $P(z) = 2z^3 - 4z + 1$ contained in (a) $|z| \le \frac{1}{4}$, (b) $|z| \le 1$, and (c) $|z| \le 3$.

17. Use the geometrical interpretation of the restricted argument principle to show that $f(z) = z - 2i + \exp(-z)$ has no zeros in $|z - i| \le 1$, one zero in $|z - i| \le 2$, and two zeros in $|z - i| \le 3$.

18. Given that $f(z) = z\exp(z) - 2z^5 + iz + 3i$, use the geometrical interpretation of the restricted argument principle to determine the number of zeros of $f(z)$ in (a) $|z| \le \frac{1}{4}$, (b) $|z| \le \frac{1}{2}$, (c) $|z| \le 1$, and (d) $|z| = \frac{3}{2}$.



CHAPTER 14
TECHNOLOGY PROJECTS

The integral of a complex function $f(z)$ along a path Γ_{AB} from point A to point B, on which f has no singularities, is simply a line integral of $f(z)$ along Γ from A to B with respect to arc length. The complex integral can be evaluated numerically as follows. First, a general point z on the arc Γ_{AB} with its initial point A at $z = z_0$ and its final point B at $z = z_1$ is expressed parametrically as $z(t) = x(t) + iy(t)$ for the parameter t in the interval $t_0 \le t \le t_1$, with $z_0 = z(t_0)$ and $z_1 = z(t_1)$. Then, on Γ, $dz = (dx/dt + i\,dy/dt)dt$, so the required integral along Γ_{AB} is given by

$$\int_{\Gamma_{AB}} f(z)dz = \int_{t_0}^{t_1} f(z(t))(dx/dt + i\,dy/dt)dt.$$

If the path Γ is continuous, but defined in a piecewise manner along successive segments, each segment must be parametrized separately. The integral along Γ then follows by adding the integrals along each of the segments. A contour integral around a simple closed curve is obtained by parametrizing the curve (in segments if necessary) and integrating once around the curve in the counterclockwise direction. If f is not analytic, the integral of f from A to B will, in general, depend on the choice of path from A to B.

Project 1

The Numerical Evaluation of Integrals along Arcs

This project uses computer algebra to calculate the integrals of complex functions f along different arcs Γ from A to B to verify, in particular cases, that when f is analytic the result is independent of the path, though when f is not analytic the integral depends on the choice of path.

1. Let A be the point $z = 1 - 2i$ and B the point $z = 1 + 2i$. Parametrize the semicircular path Γ_1 from A to B that lies to the right of the line AB and has A and B as points on opposite ends of a diameter, and find dz on Γ_1. Parametrize the piecewise continuous straight line path Γ_2 joining A to C and C to B, where C is the point $z = 2 - 2i$, and find dz on the straight line segments AC and CB.

2. Given that $f_1(z) = z\sinh(2z)$, use computer algebra to show that $f_1(z)$ satisfies the Cauchy–Riemann equations for all z, and so is an entire function.

3. Evaluate $\int_{\Gamma_1} f_1(z)dz$ and $\int_{\Gamma_2} f_1(z)dz$ and hence show, as would be expected because f is an entire function, that the integrals are equal.

4. Given that $f_2(z) = z\bar{z}\sin z$, show by using computer algebra that f_2 is not analytic. By finding $\int_{\Gamma_1} f_2(z)dz$ and $\int_{\Gamma_2} f_2(z)dz$, show that $\int_{\Gamma_1} f_2(z)dz \neq \int_{\Gamma_2} f_2(z)dz$.

Project 2

Integrating around a Circular Arc Centered on a Simple Pole

This project uses computer algebra to examine the effect of integrating around a circular arc of arbitrarily small radius when its center is located at a simple pole of a complex function $f(z)$. This process is examined analytically in Chapter 15, where it is used in the determination of definite integrals of real functions $f(x)$ over the semi-infinite interval $0 \le f < \infty$ and the infinite interval $-\infty < x < \infty$.

1. Let Γ_α be a circular arc of radius r with its center at the point $z = 1$ that subtends an angle α at $z = 1$. Denote by θ the angle from $z = 1$ to a point on the arc, with θ measured counterclockwise from the positive real axis such that $0 \le \theta \le \alpha$. Parametrize the arc Γ_α, and find dz on this arc.

2. Given that $f(z) = \cos z/(z-1)$, use computer algebra to display the integral

$$\int_{\Gamma_\alpha} f(z)dz$$

in terms of r and the parametrization of the arc Γ_α.
3. Given that $\alpha = \pi/3$, compute the integral for $r = 0.01, 0.001$, and 0.0001, and hence estimate its limiting value as $r \to 0$.
4. Repeat Step 3, using $\alpha = 2\pi/3$.
5. Repeat Step 3, using $\alpha = \pi$.
6. Repeat Step 3, using $\alpha = 5\pi/3$.
7. Compare the results of Steps 3 through 6 with the theoretical result $\int_{\Gamma_{2\pi}} f(z)dz = 2\pi i \cos(1)$, and deduce the relationship between $\int_{\Gamma_\alpha} f(z)dz$ and $\int_{\Gamma_{2\pi}} f(z)dz$ as a function of α.

Project 3

Complex Integrals around Deformed Contours

Let a function f be analytic in a region D except at a finite number of points where it has simple poles, and let Γ_1 and Γ_2 be any two contours in D both of which contain the same poles. Then contour Γ_2 can be considered to be a deformation of contour Γ_1. The purpose of the project is to use computer algebra to verify, in particular cases, that the integral around each of these contours is the same.

1. Let contour Γ_1 be the circle $|z-1| = 4$ and contour Γ_2 be the circle $|z-2-i| = 3$. Parametrize the contours, and in each case find dz on the contour.
2. Given that $f(z) = (3z-2)/(z^2-5z+6)$, verify that the poles of $f(z)$ lie inside both Γ_1 and Γ_2. Use the results of 1 with computer algebra to find $\int_{\Gamma_1} f(z)dz$ and $\int_{\Gamma_2} f(z)dz$, and hence show that they are equal.
3. Use analysis to find $\int_{\Gamma_1} f(z)dz$, and so confirm the results obtained in 2.
4. Parametrize the contour Γ_3 given by $|z-i| = 5$, and by using computer algebra to integrate around it in the clockwise sense, show that

$$\int_{\Gamma_1} f(z)dz = -\int_{\Gamma_3} f(z)dz.$$

Project 4

The Cauchy Integral Formula for Derivatives

The purpose of this project is to use computer algebra to verify the Cauchy integral formula for derivatives.

1. Parametrize the contour Γ formed by the circle $|z| = 2$ and find dz on Γ.
2. Given that $f(z) = z^2 + 3z - 7$, use computer algebra with the Cauchy integral formula to find $f(1)$.
3. Given that $f(z) = e^z(z^3 + 2z - 1)$, use computer algebra with the Cauchy integral formula for derivatives to find $f^{(2)}(1)$, and check the result by differentiation.

Projects 5–7

The Number of Zeros of a Polynomial in Each Quadrant of the *z*-Plane

Let a polynomial $P(z)$ be nonvanishing on a simple closed contour Γ, and let the total number of zeros of $P(z)$ inside Γ be N when multiplicity is counted, so that if a zero $z = a$ is repeated m times it has multiplicity m. Then

$$\frac{1}{2\pi i}\int_\Gamma \frac{P'(z)}{P(z)}dz = N.$$

The proof is simple, because if $P(z)$ has a zero of multiplicity m at $z = a$ inside Γ, it follows that $P(z)$ can be written $P(z) = (z-a)^m h(z)$, where $h(a) \neq 0$. Thus,

$$\frac{P'(z)}{P(z)} = \frac{m}{(z-a)} + \frac{h'(z)}{h(z)},$$

and as $P(z) \neq 0$ on Γ the expression on the right remains finite on Γ, so integrating around Γ gives

$$\int_\Gamma \frac{P'(z)}{P(z)}dz = 2\pi i m.$$

The result now follows by applying the preceding argument to each zero inside Γ and summing the multiplicities of the zeros to obtain N.

The purpose of Projects 5 through 7 is to use the foregoing result to find the number of zeros of the given polynomial that lies in each quadrant. To accomplish this, suitable finite size contours should be chosen and, where appropriate, use should be made of the properties of the zeros of polynomials contained in Theorem 1.2.

Project 5

$$P(z) = z^5 + 3z + 18.$$

Project 6

$$P(z) = z^4 + 2z + 6.$$

Project 7

$$P(z) = z^5 + 4iz + 3i.$$

Project 8

Identifying Regions Where a Polynomial Has No Zeros

The location of the zeros of polynomials is important in many problems: for example, in linear differential equations, where the solution will only be stable if no zeros lie to the right of the imaginary axis. The purpose of this project is to apply a theorem (see reference [6.2], Theorem 6.4b) that identifies a disc about a point z_0, which is not a zero of a given polynomial, inside and on which the polynomial has no zeros. This means that the reciprocal of the polynomial is an analytic function inside and on the boundary of the disc. The result is then to be verified numerically by integrating the reciprocal of the polynomial around the boundary of the disc and appealing to the Cauchy±Goursat theorem that asserts the result must be zero.

Let the polynomial

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

have real or complex coefficients, and let z_0 be any complex number that is not a zero of $P(z)$. Define the numbers $b_0, b_1, \ldots, b_n$ by

$$b_m = \frac{1}{m!}P^{(m)}(z_0), \quad b_0 = P(z_0) \neq 0,$$

where $P^{(m)}(z) = d^m P(z)/dz^m$ and $P^{(0)}(z_0) = P(z_0)$. Then if

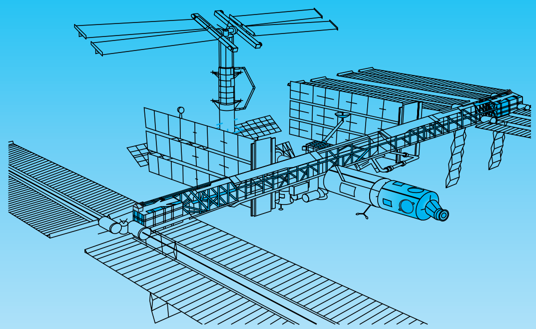
$$\rho(z_0) = \frac{1}{2}\min_{1\le m\le n}\left|\frac{b_0}{b_m}\right|^{1/m},$$

the polynomial $P(z)$ has no zeros inside or on the disc $|z - z_0| \le \rho(z_0)$.

Given $P(z) = z^4 + (1+i)z^3 + 2iz^2 + z + 2$, using a suitable value of z_0 apply the theorem to find a disc with boundary Γ inside and on which $P(z)$ has no zeros. Confirm this by using computer algebra to show numerically that, as expected from the Cauchy–Goursat theorem, $\int_\Gamma (1/P(z))dz = 0$.

This Page Intentionally Left Blank

CHAPTER 15



Laurent Series, Residues, and Contour Integration

The analytical evaluation of a general contour integral with integrand $f(z)$ depends for its success on what are called the residues at the poles of $f(z)$. The residue of a function $f(z)$ at a pole is defined in terms of a special series expansion of $f(z)$ about the pole called a Laurent series. The Laurent series represents an extension of the conventional Taylor series that is no longer applicable when an expansion of $f(z)$ is required about a singular point. Various ways of obtaining Laurent series are described, and it is shown how a contour integral is related to the residues of the integrand $f(z)$ that lie either inside or on the contour of integration. Different types of contour integral are evaluated and integration around a branch point of $f(z)$ is considered.

15.1 Complex Power Series and Taylor Series

Before introducing complex power series and discussing their convergence, it is necessary to recall the definition of a sequence. A **sequence** of real or complex numbers, or of functions, is a set of such objects arranged in a specific order, so that changing the order changes the sequence. It is conventional to enclose the terms of a sequence in brackets by writing $\{\ldots\}$. Typical examples of sequences are

$$\left\{1, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \frac{1}{5!}, \frac{1}{6!}\right\}, \quad \text{a } \textit{finite} \text{ sequence of real numbers}$$

$$\left\{\frac{1}{(1+i)}, \frac{1}{(1+i)^2}, \frac{1}{(1+i)^3}, \ldots, \frac{1}{(1+i)^n}, \ldots\right\}, \quad \text{an } \textit{infinite} \text{ sequence of complex numbers}$$

$$\left\{z, -\frac{z^3}{3!}, \frac{z^5}{5!}, -\frac{z^7}{7!}, \frac{z^9}{9!}, \ldots, (-1)^{n+1}\frac{z^{2n-1}}{(2n-1)!}, \ldots\right\}, \quad \text{an } \textit{infinite} \text{ sequence of powers of } z.$$

When working with sequences the expressions *finite* and *infinite* are used to describe to the number of terms in a sequence, and not the magnitude of any of its terms. In what follows our main concern will be with infinite sequences.

As the terms of a sequence occur in a specific order, they can be numbered sequentially like $u_1, u_2, u_3, \ldots,$ with the suffix indicating the position of a term in the sequence. Because of this a sequence can be considered to be a function f that assigns to each positive integer n the term $u_n = f(n)$, where u_n is called the **general term** of the sequence. A convenient abbreviated notation for a sequence $\{u_1, u_2, u_3, \ldots\}$ is $\{u_n\}_{n=1}^{\infty}$ or, equivalently, $\{f(n)\}_{n=1}^{\infty}$. In an infinite sequence the behavior of the general term u_n as $n \to \infty$ is its most important property, so when numbering the terms it is usually immaterial whether the suffix of the first term is 0 or 1, so the notation for an *infinite* sequence is often simplified to $\{u_n\}$.

To illustrate how the general term of a sequence can be defined in terms of a function with a positive integer argument, we consider the function

$$f(z) = \frac{1}{3^z}\sin\left\{(2z-1)\frac{\pi}{2}\right\}.$$

Setting $z = n$ and $u_n = f(n)$, with $n = 1, 2, \ldots,$ we obtain

$$u_n = (-1)^{n-1}\frac{1}{3^n},$$

so the infinite sequence with u_n as its general term becomes

$$\{u_n\}_{n=1}^{\infty} = \left\{\frac{1}{3}, -\frac{1}{3^2}, \frac{1}{3^3}, \ldots\right\} \text{ or, more simply, } \left\{(-1)^{n-1}\frac{1}{3^n}\right\}.$$

sequences, series, and *n*th partial sum

To understand the connection between infinite sequences and infinite series, let $s_n = u_1 + u_2 + \cdots + u_n$ be the sum of the first n terms of the infinite series $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots$. Then the sum of the series will be determined by the behavior of s_n as $n \to \infty$. The sum s_n is called the ***n*th partial sum** of the series, and when the terms of the series involve powers of the complex number z the nth partial sum will become a function of z, written $s_n(z)$. For any fixed z and n the function $s_n(z)$ will have a finite value. An infinite series $S(z)$ with the nth partial sum $s_n(z)$ will be said to **converge** to the value L when $z = z_0$ if, as $n \to \infty$, $\lim_{n\to\infty} s_n(z) = L$. If for some z_0 this limit is not defined, or if it is infinite, the series will be said to be **divergent**, or to **diverge** when $z = z_0$. Determining the **convergence** of an infinite power series involves finding the region in the z-plane where $\lim_{n\to\infty} s_n(z)$ is finite.

The tests for convergence that will be introduced later are applicable to the most commonly occurring types of series involving powers of z, and although they determine the *region* in the z-plane where the series converges, they do *not* determine the sum of the series.

A complex sequence $\{u_n\}$ is said to be **bounded** if some positive constant M exists such that $|u_n| < M$ for all positive integers n, and if this condition is not satisfied the sequence is said to be **unbounded**.

These ideas can be illustrated by considering the complex sequence $\{\frac{1}{6} + (-1)^n(\frac{n}{n^2+1})i\}$ that is seen to be bounded by 1 (not the sharpest bound), because the modulus of every term is less than 1. A simple example of an unbounded complex sequence is $\{ni^n\}$.

convergence, divergence, cluster points, and neighborhoods

A point α is called a **cluster point**, or a **point of accumulation**, of a sequence $\{u_n\}$ if every circle with its center at α, from which the point α itself has been deleted, contains infinitely many points of the sequence. The interior of a circle with its center at α is called a **neighborhood** of α, and a circle from which the single point α at its center has been removed is called a **deleted neighborhood** of α. A sequence

$\{u_n\}$ may have one or more cluster points, or possibly none, but when a cluster point α exists it is not necessarily a member of the sequence.

It is not difficult to see that the sequence $\{\frac{1}{6} + (-1)^n(\frac{n}{n^2+1})i\}$ only has a single cluster point at $\frac{1}{6}$, and that in this case no member of the sequence is equal to $\frac{1}{6}$. This means that however small a circle is drawn around the point $\frac{1}{6}$, infinitely many terms of the sequence will lie inside it and only a finite number will lie outside it, and no member of the sequence will lie at the center of the circle. Consequently, all but a finite number of terms of the sequence will be contained in *any* deleted neighborhood of the point $\frac{1}{6}$.

The most important type of sequence $\{u_n\}$ is one with only a single cluster point L, called the **limit** of the sequence and written

$$\lim_{n\to\infty} z_n = L.$$

A sequence with this property is said to **converge** to the limit L, and a sequence that does not converge is said to be **divergent**.

An example of a convergent infinite sequence is $\{\frac{1}{6} + (-1)^n(\frac{n}{n^2+1})i\}$, because this has a single cluster point at $\frac{1}{6}$, and so

$$\lim_{n\to\infty}\left\{\frac{1}{6} + (-1)^n\left(\frac{n}{n^2+1}\right)i\right\} = \frac{1}{6}.$$

limits and convergence of complex series

When expressed in words, the definition of the limit of a sequence says that a sequence $\{u_n\}$ will have a limit L if, and only if, however small we take the radius of a circle with its center at L, there are infinitely many terms of the sequence inside the circle and only finitely many outside it. The limit L of a convergent sequence $\{u_n\}$ is illustrated in Fig. 15.1 where the deleted neighborhood of L is indicated by the interior of the circle centered on L with an arbitrarily small radius ε. Finitely many points of $\{u_n\}$ lie outside this circle and infinitely many lie inside it, and although in the limit as $n \to \infty$, $u_n \to L$, it is not necessary that L be a member of the sequence.

A more precise definition of the limit of a convergent complex sequence can be formulated as follows.

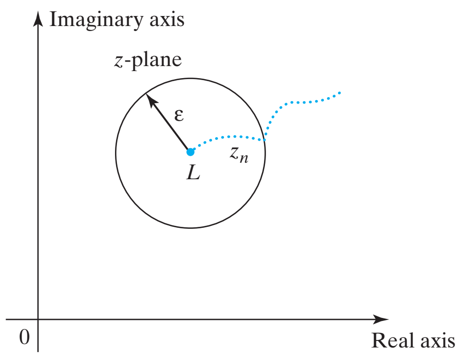


FIGURE 15.1 A convergent complex sequence $\{z_n\}$ with limit L.

A Convergent Sequence

A complex sequence $\{z_n\}$ will be said to **converge** to the **limit** L if for every arbitrarily small number $\varepsilon > 0$ a positive integer N can be found such that

$$|z_n - L| < \varepsilon \quad \text{for all } n > N.$$

As this definition of the limit of a convergent sequence applies to real and complex sequences, when $L = L_1 + iL_2$ is complex the definition implies that if $z_n = u_n + iv_n$, then

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}(u_n + iv_n) = \lim_{n\to\infty} u_n + i \lim_{n\to\infty} v_n = L_1 + iL_2,$$

and so

$$\lim_{n\to\infty} u_n = L_1 \quad \text{and} \quad \lim_{n\to\infty} v_n = L_2.$$

A formal proof of this result involves using the more precise definition of the limit of a convergent sequence given earlier, but as the proof is straightforward the details are left as an exercise.

EXAMPLE 15.1 Find any cluster points that belong to the following sequences and, where appropriate, find the limit of the sequence.

$$\text{(a)}\ \left\{1 + (-1)^n + \frac{1}{n!}\right\}, \quad \text{(b)}\ \{n\}, \quad \text{(c)}\ \left\{\left(\frac{2n+1}{n}\right) + i\left(\frac{3n^2-1}{n^2}\right)\right\}.$$

Solution

(a) As n increases, the first two terms combine to give either 0 or 2, according as n is odd or even, and the third term tends to zero as $n \to \infty$. Thus, as n increases, so terms of the sequence cluster ever closer around the numbers 0 and 2, showing that this sequence has two cluster points. Any small circle drawn around one of the cluster points that excludes the other will contain infinitely many points of the sequence, though infinitely many will remain outside it. This sequence is bounded but has no limit because it has more than one cluster point, and so it is divergent.

(b) It is clear by inspection that this sequence is unbounded and has no cluster points, so it is divergent.

(c) Setting

$$z_n = \left(\frac{2n+1}{n}\right) + i\left(\frac{3n^2-1}{n^2}\right),$$

we see that

$$\lim_{n\to\infty}\left(\frac{2n+1}{n}\right) = 2 \quad \text{and} \quad \lim_{n\to\infty}\left(\frac{3n^2-1}{n^2}\right) = 3,$$

so this sequence is bounded and only has the single cluster point $2 + 3i$. Thus, the sequence converges to the limit $2 + 3i$. This limit is *not* a member of the sequence, because for no finite n is it true that $z_n = 2 + 3i$. ■

The foregoing definition of convergence makes use of the limit of the sequence, but this is not always easy to find, so it is desirable to have a test for convergence that

does not involve the limit itself. This is made possible by introducing the concept of a Cauchy sequence. A sequence $\{z_n\}$ is called a **Cauchy sequence** if for any arbitrarily small number $\varepsilon > 0$ it is always possible to find an integer N, usually depending on ε, such that $|z_m - z_n| < \varepsilon$ for all $m > n > N$.

In effect, a Cauchy sequence is one with the property that, however small the number ε is chosen, it is always possible to find a large positive integer N such that the modulus of the difference between *any* two terms of the sequence with index greater than N will always be less than ε.

Although we omit the proof, it can be shown that a Cauchy sequence $\{z_n\}$ must converge to a limit. This result forms our next theorem.

THEOREM 15.1

Cauchy convergence principle for sequences A sequence $\{z_n\}$ converges if, and only if, for any arbitrary small number $\varepsilon > 0$ it is possible to find an integer N depending on ε such that $|z_m - z_n| < \varepsilon$ for all $m > n > N$. ■

EXAMPLE 15.2

Use Theorem 15.1 to prove the convergence of the sequence $\{(\cos n\pi)/n\}$.

Solution Setting $z_n = (\cos n\pi)/n$ we have

$$|z_m - z_n| = \left|\frac{\cos m\pi}{m} - \frac{\cos n\pi}{n}\right| \leq \frac{n|\cos m\pi| + m|\cos n\pi|}{mn} = \frac{m+n}{mn}.$$

Now, if $m > n > N$, then

$$\frac{m+n}{mn} < \frac{2m}{mn} = \frac{2}{n} < \frac{2}{N},$$

so

$$|z_m - z_n| < \frac{2}{N}.$$

Consequently, for any arbitrary $\varepsilon > 0$, provided N is chosen such that $2/N < \varepsilon$, the conditions of Theorem 15.1 are satisfied and the sequence converges. In this case the convergence of the sequence to the limit 0 is obvious, because $\cos n\pi = (-1)^n$, so the general term of the sequence is simply $(-1)^n/n$. ■

It has already been shown that the sum of an infinite series can be regarded as the limit of the operation of sequentially adding the terms of an infinite sequence. Consequently, if the sequence of partial sums has a limit L, this must be the limit of the infinite series formed in this manner. If the infinite series involves powers of z, and so is a **power series**, its convergence or divergence will depend on z. For a power series to be useful it will be necessary to determine the region in the z-plane where it converges.

The proofs of the following results for complex series closely parallel the corresponding results for real series, so the results will merely be stated.

THEOREM 15.2

Limit of complex series Let $z_n = u_n + iv_n$, and denote the nth partial sum of the series $\sum_{n=1}^{\infty} z_n$ by

$$s_n = \sum_{m=1}^{n} u_m + i\sum_{m=1}^{n} v_m.$$

Then a necessary and sufficient condition for the series to converge is that the sequences $\{\sum_{m=1}^{n} u_m\}$ and $\{\sum_{m=1}^{n} v_m\}$ converge as $n \to \infty$. When this is true, if $\lim_{n\to\infty} \sum_{m=1}^{n} u_m = L_1$ and $\lim_{n\to\infty} \sum_{m=1}^{n} v_m = L_2$, then $\lim_{n\to\infty} s_n = L_1 + iL_2$. ■

THEOREM 15.3

A necessary condition satisfied by convergent series If the series $\sum_{n=1}^{\infty} z_n$ converges, then $\lim_{n\to\infty} z_n = 0$. ■

The main use of this theorem is to establish the *divergence* of a series, because if $\lim_{n\to\infty} z_n \neq 0$ the series *cannot* converge. The theorem provides no information about convergence, because the condition $\lim_{n\to\infty} z_n = 0$ is *not* sufficient to ensure the convergence of a series. This is easily seen by considering the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, because setting $z_n = \frac{1}{n}$ we see that $\lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{1}{n} = 0$, but the series is known to diverge.

EXAMPLE 15.3

Show the series $\sum_{n=1}^{\infty} \frac{(n^2-2ni)}{3n+4}$ is divergent.

Solution The general term is $z_n = \frac{n^2-2ni}{3n+4}$. However, $\lim_{n\to\infty} z_n = \frac{n}{3} - \frac{2i}{3} \neq 0$, so it follows from Theorem 15.3 that the series is divergent. ■

Convergence Tests for Complex Series

The relationship that exists between sequences and series allows the Cauchy convergence principle for sequences to be reinterpreted for series in the following form.

THEOREM 15.4

Cauchy convergence principle for series The infinite series $\sum_{n=1}^{\infty} z_n$ is convergent if, and only if, for every arbitrarily small number $\varepsilon > 0$ a positive integer N can be found depending on ε such that

$$|z_{n+1} + z_{n+2} + \cdots + z_{n+r}| < \varepsilon \quad \text{for every } n > N \text{ and } r = 1, 2, \ldots.$$ ■

Expressed in words, this theorem says that if an infinite series is convergent, then, however small ε, it is always possible to find a positive integer N such that the modulus of the sum of *any* number of consecutive terms starting with index greater than N will be less than ε. If the series is written $z_1 + z_2 + \cdots + z_n + R_n$, where $R_n = z_{n+1} + z_{n+2} + z_{n+3} + \cdots = \sum_{m=n+1}^{\infty} z_m$ is called the **remainder** after n terms, the theorem asserts that $|R_N| < \varepsilon$. In practical terms this means that if the infinite series is approximated by the sum of the first N terms, the error involved cannot exceed ε.

A series $\sum_{n=1}^{\infty} z_n$ with the property that the sum of the moduli of z_n is also convergent is said to be **absolutely convergent**. Thus, the series $\sum_{n=1}^{\infty} z_n$ is *absolutely convergent* if the series

$$\sum_{n=1}^{\infty} |z_n| = |z_1| + |z_2| + \cdots$$

is convergent.

If, however, the series $\sum_{n=1}^{\infty} z_n$ is convergent but the series $\sum_{n=1}^{\infty} |z_n| = |z_1| + |z_2| + \cdots$ is divergent, the series $\sum_{n=1}^{\infty} z_n$ is said to be **conditionally convergent**. Absolute and conditional convergence are most easily illustrated by considering the real series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n+1}\frac{1}{n} + \cdots$. It is known from elementary calculus that the sum of this series is ln 2, and so it is convergent. However, the sum of the absolute values is the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$, which is known to be divergent, so the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n+1}\frac{1}{n} + \cdots$ is conditionally convergent.

One direct consequence of Theorem 15.4 is that absolute convergence *implies* convergence. Another consequence of the theorem is the following result, which we state in the form of a theorem.

THEOREM 15.5

a simple comparison test for convergence

Comparison test for convergence Let a series $\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots$ be given, and let the series $\sum_{n=1}^{\infty} b_n$ with nonnegative terms b_n be convergent and such that $|z_n| \le b_n$ for $n = 1, 2, \ldots$. Then the series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent.

Proof As the series $\sum_{n=1}^{\infty} b_n$ is convergent by hypothesis, for any $\varepsilon > 0$ there exists an integer N such that $b_{n+1} + b_{n+2} + \cdots + b_{n+r} < \varepsilon$ for all $n > N$ and $r = 1, 2, \ldots$.

As $|z_n| < b_n$ for every n, it follows that

$$|z_{n+1}| + |z_{n+2}| + \cdots + |z_{n+r}| \le b_{n+1} + b_{n+2} + \cdots + b_{n+r} < \varepsilon,$$

so by Theorem 15.4 the series $\sum_{n=1}^{\infty} |z_n| = |z_1| + |z_2| + \cdots$ converges, showing the series $\sum_{n=1}^{\infty} z_n$ to be absolutely convergent. ■

Several tests for convergence use, for purposes of comparison, the infinite **geometric series**

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots,$$

which an elementary argument shows converges to the sum $1/(1-r)$ if $|r| < 1$, and diverges if $|r| \ge 1$.

Because the convergence of an infinite geometric series depends on the magnitude of $|r|$, convergence tests based on a comparison with the geometric series lead to tests for *absolute convergence*. When these tests are applied to real series with positive terms they become tests for *convergence*. The most important and useful of these tests are the ratio and nth root tests.

THEOREM 15.6

the ratio test for convergence

The ratio test Let a series $\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots$, in which no term is zero, be such that

$$\lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| = L.$$

Then the absolute convergence or divergence of the series is determined by the following conditions:

(i) If $L < 1$, the series converges absolutely

(ii) If $L > 1$, the series diverges

(iii) If $L = 1$, the test fails and no conclusion can be drawn about the convergence of the series.

Proof Suppose that $|z_{n+1}/z_n| \leq \alpha < 1$ for n greater than some positive integer N. Then $|z_{n+1}| \leq |z_n|$ and we have

$$|z_{N+2}| \leq \alpha|z_{N+1}|, \quad |z_{N+3}| \leq \alpha|z_{N+2}| \leq \alpha^2|z_{N+1}|, \ldots,$$

leading to the general result $|z_{N+r}| \leq \alpha^{r-1}|z_{N+1}|$.

If R_N is the remainder of the series after N terms, this last result allows its modulus to be estimated by

$$|R_N| \leq |z_{N+1}| + |z_{N+2}| + |z_{n+3}| + \cdots \leq |z_{N+1}|(1 + \alpha + \alpha^2 + \alpha^3 + \cdots).$$

The bracketed geometric series converges when $\alpha < 1$, so as $|R_N|$ is bounded the series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent. Conversely, the bracketed geometric series is divergent if $|\alpha| > 1$, showing that then the series $\sum_{n=1}^{\infty} z_n$ must be divergent. If $\alpha = 1$ the test fails in the sense that it provides no information about the convergence of the series. The statement of the theorem follows directly from these conclusions. ■

It is important to recognize that the real constant α in the ratio test must be strictly less than 1. This is essential in order to exclude series such as the harmonic series that, although divergent, have a limiting ratio $|z_{n+1}/z_n|$ that approaches arbitrarily close to 1 as $n \to \infty$.

EXAMPLE 15.4 Apply the ratio test to the series

$$\text{(a)} \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n!}{n^n}, \quad \text{(b)} \sum_{n=1}^{\infty}\frac{i^n}{(3n+2)^2}, \quad \text{and} \quad \text{(c)} \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^{n+1}}{n+2}.$$

Solution

(a) Setting $z_n = (-1)^{n+1}n!/n^n$ we find that

$$\left|\frac{z_{n+1}}{z_n}\right| = \frac{(n+1)!n^n}{(n+1)^n n!} = \left(1 + \frac{1}{n}\right)^{-n},$$

but from from Table 15.1 it is seen that $\lim_{n\to\infty}(1 + \frac{1}{n})^{-n} = 1/e$, so

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = \frac{1}{e} < 1.$$

Thus, as $L = 1/e < 1$, it follows from the ratio test that the series is absolutely convergent.

(b) Setting $z_n = i^n/(3n+2)^2$ we find that

$$\left|\frac{z_{n+1}}{z_n}\right| = \left(\frac{3n+2}{3n+5}\right)^2,$$

so

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty}\left(\frac{3n+2}{3n+5}\right)^2 = 1.$$

In this case the limit $L = 1$, so the ratio test fails. In fact, the series is absolutely convergent, as may be seen by comparison with the convergent series $\sum_{n=1}^{\infty} 1/n^2$ given in Table 15.1.

TABLE 15.1 Some Useful Comparison Series and Limits

1.	$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \cdots + \frac{1}{n!} + \cdots = e$	(convergent)
2.	$\sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} + \cdots = 1/e$	(absolutely convergent)
3.	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^{n+1} \frac{1}{n} + \cdots = \ln 2$	(conditionally convergent)
4.	$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$	(divergent; this is the *harmonic* series)
5.	$\sum_{n=0}^{\infty} \alpha^n = 1 + \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^n + \cdots = \frac{1}{1-\alpha}$	(convergent for $\lvert\alpha\rvert < 1$ and divergent for $\lvert\alpha\rvert \geq 1$; this is the *geometric series*)
6.	$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots = \frac{\pi^2}{6}$	(convergent)
7.	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots + (-1)^{n+1} \frac{1}{n^2} + \cdots = \frac{\pi^2}{12}$	(absolutely convergent)
8.	$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} = 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \cdots + \frac{1}{n^\alpha} + \cdots$	(convergent if $\alpha > 1$ and divergent if $0 < \alpha \leq 1$; this is the *harmonic series* of order α)
9.	$\lim_{n\to\infty} \left(1 + \frac{\alpha}{n}\right)^n = e^\alpha$	
10.	$\lim_{n\to\infty} \sqrt[n]{n} = 1$	
11.	$\lim_{n\to\infty} \frac{n!}{n^n} = 0$	
12.	$\lim_{n\to\infty} \frac{n^n}{n!} = \infty$	

(c) Setting $z_n = (-1)^{n+1} \frac{2^{n+1}}{n+2}$, we have

$$\left|\frac{z_{n+1}}{z_n}\right| = 2\left(\frac{n+2}{n+3}\right),$$

so

$$\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = 2 \lim_{n\to\infty} \left(\frac{n+2}{n+3}\right) = 2,$$

showing that $L = 2$, but as $L > 1$ the ratio test shows this series to be divergent. ■

The nth root test can be established in a manner similar to that of the ratio test, so the details of its proof will be omitted.

THEOREM 15.7

the nth root test for convergence

The nth root test for convergence Let a series $\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots$, in which no term is zero, be such that

$$\lim_{n\to\infty} \sqrt[n]{z_n} = L.$$

Then the absolute convergence and divergence of the series is determined by the following conditions:

(i) If $L < 1$, the series converges absolutely
(ii) If $L > 1$, the series diverges
(iii) If $L = 1$ the test fails, and no conclusion can be drawn about the convergence of the series. ■

EXAMPLE 15.5 Find conditions on the real constant α in order that the series

$$\sum_{n=1}^{\infty}\left(\frac{\alpha n}{\alpha n+1}\right)^{n^2} i^n$$

is absolutely convergent.

Solution Setting $z_n = (\frac{\alpha n}{\alpha n+1})^{n^2} i^n$ we have

$$\sqrt[n]{\left|\left(\frac{\alpha n}{\alpha n+1}\right)^{n^2} i^n\right|} = \left|\frac{\alpha n}{\alpha n+1}\right|^n = 1\Big/\left(1+\left(\frac{1}{\alpha}\right)\left(\frac{1}{n}\right)\right)^n,$$

and making use of a limit in Table 15.1 we see that

$$L = \lim_{n\to\infty}\sqrt[n]{|z_n|} = 1/e^{1/\alpha} = e^{-1/\alpha}.$$

As $L < 1$ if $\alpha > 0$ and $L > 1$ if $\alpha < 0$, the nth root test shows that the series is absolutely convergent if $\alpha > 0$ and divergent if $\alpha < 0$. ■

Complex Power Series and Circles of Convergence

A series of the form

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + a_n(z-z_0)^n + \cdots, \tag{1}$$

complex power series

in which the a_n, z, and z_0 are complex, is called a **complex series** in powers of $z - z_0$, or simply a **complex power series**, expanded about the point z_0. In complex power series the complex number z_0 is often called the **center** of the series. The convergence of such series depends on the **coefficients** of the series, that is, the numbers a_n, on the complex variable z, and on the point z_0 about which the series is expanded. To determine the conditions to be imposed on a_n, z, and z_0 in order to ensure convergence, we apply either the ratio test or the nth root test to the nth term $a_n(z-z_0)^n$ of the complex power series in (1).

An application of the ratio test shows that the series will be convergent if

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}(z-z_0)^{n+1}}{a_n(z-z_0)^n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right||z-z_0| < 1,$$

and this is equivalent to the condition

$$|z - z_0| < \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = R, \tag{2}$$

where the number

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \tag{3}$$

is called the **radius of convergence** of the complex power series in (1). In terms of R the condition for absolute convergence in (2) becomes

$$|z - z_0| < R, \tag{4}$$

showing that the series is absolutely convergent for all z inside a circle of radius R with its center at the point z_0.

A similar argument applied to the complex power series in (1), but this time using the nth root test, gives

$$L = \lim_{n\to\infty}\sqrt[n]{|a_n(z - z_0)^n|} = \lim_{n\to\infty}\sqrt[n]{|a_n|}|z - z_0| < 1,$$

showing that the series will be absolutely convergent if

$$|z - z_0| < R, \quad \text{where } R = 1/\lim_{n\to\infty}\sqrt[n]{|a_n|}. \tag{5}$$

radius and circle of convergence

Summarizing these results, we see that the **radius of convergence** R of the power series in (1) and its associated **circle of convergence**, that is, the circle $|z - z_0| < R$, can be found either from

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|, \quad \text{with } |z - z_0| < R, \tag{6}$$

or from

$$R = 1/\lim_{n\to\infty}\sqrt[n]{|a_n|}, \quad \text{with } |z - z_0| < R. \tag{7}$$

The choice of which one of these results to use in practice is determined by whichever limit is the simpler to evaluate.

EXAMPLE 15.6

Find the radius and circle of convergence of the power series

$$\text{(a)} \sum_{n=1}^{\infty}\frac{(z-i)^n}{n} \quad \text{and} \quad \text{(b)} \sum_{n=1}^{\infty}\frac{n(5+2i)^n}{3^n}(z-1)^n.$$

Solution

(a) In this case result (6) is simpler to use, so setting $a_n = 1/n$ and $z_0 = i$ gives

$$R = \lim_{n\to\infty}\left|\frac{n+1}{n}\right| = 1.$$

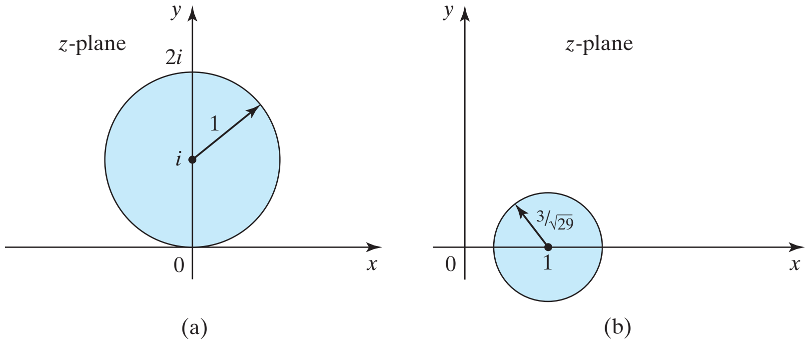


FIGURE 15.2 Circles of convergence.

So the radius of convergence is $R = 1$ and the circle of convergence is $|z - i| < 1$. This is illustrated in Fig. 15.2a.

(b) Here result (7) is simpler to use, so setting $a_n = \frac{n(5+2i)^n}{3^n}$ and $z_0 = 1$ gives

$$R = 1/\lim_{n\to\infty} \sqrt[n]{\left|\frac{n(5+2i)^n}{3^n}\right|} = \frac{3}{|5+2i|} \lim_{n\to\infty} (1/\sqrt[n]{n}) = \frac{3}{\sqrt{29}},$$

where when determining the limit use has been made of entry 10 in Table 15.1. This series converges in a circle of radius $R = 3/\sqrt{29}$ with its center at the point $z_0 = 1$, as shown in Fig. 15.2b.

Power series define functions, so it is necessary to know if they possess the property of continuity, and whether they can be added, multiplied, differentiated, and integrated. Furthermore, as each partial sum $s_n(z)$ of a power series is a polynomial in z, and so is an analytic function, it is necessary to know if the power series itself is also an analytic function. The answer to each of these questions is in the affirmative, and they form the substance of the next theorem. ■

THEOREM 15.8

important properties of complex power series

Properties of power series Power series with finite circles of convergence possesses the following properties:

(i) A power series represents a continuous function at each point inside its circle of convergence.

(ii) If two power series expanded about the same point have the same circle of convergence D and the same sum at each point of D, then they are identical.

(iii) If two power series with sums $f(z)$ and $g(z)$ and circles of convergence D_1 and D_2 are added or subtracted term by term, the result is a power series that converges to the sum $f(z) \pm g(z)$ with a circle of convergence that is at least equal to the largest circle that can be drawn in the region common to D_1 and D_2.

(iv) If two power series with sums $f(z)$ and $g(z)$ and circles of convergence D_1 and D_2 are multiplied, the result is a power series that converges to the product $f(z)g(z)$ with a circle of convergence that is at least equal to the largest circle that can be drawn in the region common to D_1 and D_2.

(v) If a power series with the sum $f(z)$ and a circle of convergence D are differentiated term by term, the result is a power series that converges to $f'(z)$ at each point in D.

(vi) If a power series with the sum $f(z)$ and a circle of convergence D is integrated term by term, the result is a sum that converges to $\int f(z)dz$ at each point in D.

(vii) A power series with a circle of convergence D is an analytic function in D.

Proof Only results with proofs that are straightforward will be outlined in order to avoid introducing unnecessary complication.

(i) It will be sufficient to prove that a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with a nonzero radius of convergence R and circle of convergence Γ represents a continuous function of z at every point inside Γ. This is because if the power series is expanded about a point z_0 instead of the origin, the change of variable $w = z - z_0$ will reduce it to this case. Continuity will be proved if we can show that for any point ζ inside Γ and for any given $\varepsilon > 0$, it follows that $|f(z) - f(\zeta)| < \varepsilon$ for all z inside Γ such that $|z - \zeta| < \delta$.

Set $f(z) = S_N(z) + R_N(z)$, where $S_N(z) = \sum_{n=0}^{N} a_n z^n$ and the remainder $R_N(z) = \sum_{n=N}^{\infty} a_n z^n$. Let D be the interior and boundary of any circle C with its center at the origin and its radius $r < R$. Then to proceed further it is necessary to anticipate the result of Theorem 15.11 by using the uniform convergence of the power series in C to guarantee the existence of a positive integer $N = N(\varepsilon)$ such that $|f(z) - S_N(z)| < \frac{1}{3}\varepsilon$ for all z in D. The series $S_N(z)$ is simply a polynomial in z, so it follows that it must be a continuous function of z. Consequently, with this value of N, there must be a $\delta > 0$ such that $|S_N(z) - S_N(\zeta)| < \frac{1}{3}\varepsilon$ when $|z - \zeta| < \delta$.

Then, for all z in D such that $|z - \zeta| < \delta$, we can write

$$\begin{aligned} |f(z) - f(\zeta)| &= |f(z) - S_N(z) + S_N(z) - S_N(\zeta) + S_N(\zeta) - f(\zeta)| \\ &\le |f(z) - S_N(z)| + |S_N(z) - S_N(\zeta)| + |S_N(\zeta) - f(\zeta)| \\ &< \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon, \end{aligned}$$

so the continuity of the power series at all points of D has been established. The statement of the theorem now follows because C is any circle with its center at the origin with $r < R$.

(ii) As with (i), it will be sufficient to consider the two power series expanded about the origin $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$, each with the same circle of convergence D throughout which each converges to the same sum. Then for all z in D we have, by hypothesis,

$$a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots = b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots.$$

By (i) the sums are continuous at $z = 0$, so $a_0 = b_0$. Cancelling these terms and removing a factor z, we arrive at the result

$$a_1 + a_2 z + a_3 z^2 + \cdots = b_1 + b_2 z + b_3 z^2 + \cdots,$$

and a repetition of the argument shows that $a_1 = b_1$. Continuing this process by induction, we conclude that $a_n = b_n$ for $n = 0, 1, 2, \ldots$, so the uniqueness of the power series has been proved.

(iii) The result follows by adding or subtracting the two nth partial sums and proceeding to the limit as $n \to \infty$.

(iv) Though not difficult, the proof of this result is lengthy and so will be omitted.

(v) Let the circle of convergence of $f(z)$ be D. The convergence of the differentiated series to $f'(z)$, and the demonstration that the differentiated series has the

same circle of convergence D, follows by using term-by-term differentiation and applying the ratio test to the result.

(vi) Let the circle of convergence of $f(z)$ be D. Then the convergence of the integrated series to $\int f(z)dz$, and the demonstration that the integrated series has the same circle of convergence D, follows by using term-by-term integration and applying the ratio test to the result.

(vii) The details of the proof of this result are complicated and so will be omitted. ■

Complex power series arise in many different ways, the most frequent of which is in the form of Taylor series expansions of functions. The **Taylor series** expansion of an analytic function f about the point z_0 takes the same form as the Taylor series for a function of a real variable, though the derivation of the result is different.

THEOREM 15.9

the complex form of Taylor's series

Taylor's theorem Let $f(z)$ be an analytic function of z at the point z_0, and let it also be analytic inside a circle C given by $|z - z_0| = r$ that forms a neighborhood of z_0. Then there exists a power series

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n$$

with coefficients a_n determined by the formula

$$a_n = \frac{f^{(n)}(z_0)}{n!} \quad \text{for } n = 0, 1, 2, \ldots,$$

which converges to $f(\zeta)$ for every ζ inside the circle C and is such that

$$f(\zeta) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(\zeta - z_0)^n.$$

Proof Without loss of generality the result will be proved for $z_0 = 0$, because a change of origin extends the result to the case $z_0 \neq 0$. The proof is based on the Cauchy integral formula for derivatives and makes use of the identity

$$\frac{f(z)}{z-\zeta} = \frac{f(z)}{z} + \zeta\frac{f(z)}{z^2} + \cdots + \zeta^{n-1}\frac{f(z)}{z^n} + \zeta^n\frac{f(z)}{(z-\zeta)z^n},$$

which is easily verified for $z \neq 0$ and $z \neq \zeta$.

As $z_0 = 0$, the circle C in the theorem becomes the circle $|z| = r$. If we multiply the preceding identity by $1/(2\pi i)$ and integrate around any positively oriented circle Γ inside C with its center at the origin and radius $\rho (0 < \rho < r)$, it follows from the analytic nature of $f(\zeta)$ and the Cauchy integral formula for derivatives that

$$f(\zeta) = f(0) + \zeta f'(0) + \zeta^2\frac{f''(0)}{2!} + \cdots + \zeta^{n-1}\frac{f^{(n-1)}(0)}{(n-1)!} + \frac{\zeta^n}{2\pi i}\int_{\Gamma}\frac{f(z)}{(z-\zeta)z^n}dz.$$

This is **Taylor's theorem with a remainder**, where the last term is the remainder R_n after n terms. The proof will be complete once we have shown that $R_n \to 0$ as $n \to \infty$. From the maximum modulus theorem we know that a number $M > 0$ can be found such that $|f(z)| < M$ for all z inside the circle Γ, so on Γ

$$\left|\frac{\zeta^n f(z)}{(z-\zeta)z^n}\right| \le \frac{M}{|z-\zeta|}\left|\frac{\zeta}{z}\right|^n.$$

Using this result in R_n leads to the estimate

$$|R_n| = \left|\frac{1}{2\pi i}\int_\Gamma \frac{\zeta^n f(z)}{(z-\zeta)z^n}dx\right| \le \frac{1}{2\pi}\frac{M}{|z-\zeta|}\left|\frac{\zeta}{z}\right|^n 2\pi\rho = \frac{M\rho}{|z-\zeta|}\left|\frac{\zeta}{z}\right|^n.$$

Now as z lies on Γ and ζ is inside Γ, it follows that $|\zeta/z| < 1$, and so $|\zeta/z|^n \to 0$ as $n \to \infty$. The result $|z| = \rho$ allows the elementary inequality $|z-\zeta| \ge ||z| - |\zeta||$ to be written as $|z-\zeta| \ge |\rho - |\zeta||$, so that the expression for $|R_n|$ becomes

$$|R_n| \le \frac{M\rho}{|\rho - |\zeta||}\left|\frac{\zeta}{z}\right|^n.$$

Finally, as $|\zeta|$ is a constant and $|\zeta/z|^n \to 0$ as $n \to \infty$, proceeding to the limit as $n \to \infty$ shows that $\lim_{n\to\infty}|R_n| = 0$, and hence that $\lim_{n\to\infty} R_n = 0$. Thus, we have proved that

$$f(\zeta) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\zeta^n$$

at all points inside Γ. As $0 < \rho < r$, this result is also true for all points inside C. The proof is complete. ■

BROOKE TAYLOR (1685–1731)
An English mathematician educated at Cambridge and whose interests extended beyond mathematics to religion and philosophy. He was responsible for the introduction into mathematics of the method of finite differences in a work published between 1715 and 1717, that also contained what is now known as "Taylor's Theorem." Taylor did not consider the convergence of his series and it was not until a century later that Cauchy provided a satisfactory convergence proof. Taylor obtained a series solution for an initial value problem for a differential equation by repeatedly differentiating the equation to find the coefficients to substitute into his series solution.

The complex power series

$$f(\zeta) = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(\zeta - z_0)^n \tag{8}$$

is called the **Taylor series** of the analytic function $f(\zeta)$ expanded about the **point** (or **center**) z_0, and when $z_0 = 0$ this becomes the **Maclaurin series** expansion of $f(\zeta)$.

The derivation of Taylor's theorem shows that the radius of convergence of the Taylor series of a function with its center at z_0 will be the radius of the largest circle centered on z_0 inside which the function is analytic.

EXAMPLE 15.7 Find the Taylor series expansion of $f(z) = \cos z$ with its center at $z_0 = c$, and hence deduce its Maclaurin series expansion.

Solution The cosine function is an entire function and so can be expanded as a power series about any center, so the resulting series will have an arbitrarily large radius of convergence.

Routine differentiation gives

$$\frac{d[\cos z]}{dz} = -\sin z,\ \frac{d^2[\cos z]}{dz^2} = -\cos z,\ \frac{d^3[\cos z]}{dz^3} = \sin z,\ \frac{d^4[\cos z]}{dz^4} = \cos z, \ldots,$$

so substituting these results in the Taylor series (8), setting $z_0 = c$, and replacing ζ by z shows the required Taylor series to be

$$\cos z = \cos c - \frac{\sin c}{1!}(z-c) - \frac{\cos c}{2!}(z-c)^2 + \frac{\sin c}{3!}(z-c)^3 + \frac{\cos c}{4!}(z-c)^4 - \cdots.$$

The cosine function is an entire function, so this series converges for all z.

The Maclaurin series for $\cos z$ is obtained from this by setting $c = 0$, when we find that

$$\cos z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots.$$ ■

It is seen from this example that the complex Maclaurin series for $\cos z$ can be obtained from the corresponding series involving the real variable x by simply replacing x by z. This result remains true in general for Taylor series of elementary functions of a real variable. Some useful results that can be obtained in this manner are listed next. Here, for completeness, the expansion of $\cos z$ has been included:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots, \qquad |z| < \infty \tag{9}$$

$$\sin z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots, \qquad |z| < \infty \tag{10}$$

$$\cos z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots, \qquad |z| < \infty \tag{11}$$

$$\mathrm{Log}(1+z) = \sum_{n=1}^{\infty}(-1)^{n+1} \frac{z^n}{n} = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots, \qquad |z| < 1 \tag{12}$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots, \qquad |z| < \infty \tag{13}$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots, \qquad |z| < \infty. \tag{14}$$

Alternative Ways of Obtaining Power Series Expansions

other ways of finding Taylor series expansions

A Taylor series is a power series, so it follows from the uniqueness of power series in Theorem 15.8 (ii) that however a power series expansion of a function $f(z)$ about a point z_0 is obtained, it must be the Taylor series expansion of the function about the

same point. This property of power series is of considerable practical importance, because it is often easier to obtain a power series expansion of a function by methods that do not require the repeated differentiations needed to find the coefficients of a Taylor series. Typical ways in which power series expansions of functions can be obtained are by substitution into known simpler series, by multiplication of series, by use of the binomial theorem, or by differentiation or integration of known simpler series. Some representative examples of these ways are given below.

Expansion by the binomial theorem and a substitution

EXAMPLE 15.8

Find the Taylor series expansion of $f(z) = (8+z)^{-1/2}$ about the point $z_0 = 1$.

Solution To introduce powers of $(z-1)$ into the expansion we write $f(z) = (8+z)^{-1/2}$ as

$$f(z) = \frac{1}{3}\frac{1}{\left[1+\frac{1}{9}(z-1)\right]^{1/2}},$$

and after setting $u = z-1$ we expand $\frac{1}{3}(1+\frac{1}{9}u)^{-1/2}$ by the binomial theorem to obtain

$$\frac{1}{3\left(1+\frac{1}{9}u\right)^{1/2}} = \frac{1}{3} - \frac{1}{54}u + \frac{1}{648}u^2 - \frac{5}{34992}u^3 + \cdots.$$

Replacing u by $z-1$ we arrive at the required Taylor series expansion about the point $z_0 = 1$:

$$\frac{1}{(8+z)^{1/2}} = \frac{1}{3} - \frac{1}{54}(z-1) + \frac{1}{648}(z-1)^2 - \frac{5}{34992}(z-1)^3 + \cdots.$$

The binomial expansion of $(1+\frac{1}{9}u)^{-1/2}$ converges for $|u/9| < 1$, so the required Taylor series converges for $|z-1| < 9$. ■

Series obtained by integration

EXAMPLE 15.9

Find the Maclaurin series expansion of Arcsin z.

Solution We start from the result

$$\arcsin z = \int \frac{dz}{(1-z^2)^{1/2}}.$$

Expanding the integrand by the binomial theorem and integrating term by term gives the power series expansion for the general function arcsin z. Confining attention to the principal branch Arcsin z for which Arcsin $0 = 0$ shows that the arbitrary integration constant is zero, so

$$\text{Arcsin } z = z + \frac{1}{6}z^3 + \frac{3}{40}z^5 + \frac{5}{112}z^7 + \cdots.$$

As the principal branch is required, we must restrict z so that $\text{Re}\{\text{Arcsin } z\} < |\pi/2|$. ■

Series obtained by using a partial fraction representation

EXAMPLE 15.10 Find the Taylor series expansion of $f(z) = 1/[(z-2)(z-3)]$ about the point $z_0 = 1$.

Solution To introduce powers of $(z-1)$ we write $f(z)$ as

$$f(z) = \frac{1}{[(z-1)-1][(z-1)-2]}$$

and set $u = z - 1$. A partial fraction expansion of the resulting expression in u gives

$$\frac{1}{(u-1)(u-2)} = \frac{1}{(1-u)} - \frac{1}{2}\frac{1}{\left(1-\frac{1}{2}u\right)}.$$

Expanding each of these terms by the binomial theorem and combining the results gives

$$\frac{1}{(u-1)(u-2)} = \frac{1}{2} + \frac{3}{4}u + \frac{7}{8}u^2 + \frac{15}{16}u^3 + \cdots.$$

Replacing u by $z-1$ shows that the required Taylor series expansion is

$$\frac{1}{(z-2)(z-3)} = \frac{1}{2} + \frac{3}{4}(z-1) + \frac{7}{8}(z-1)^2 + \frac{15}{16}(z-1)^3 + \cdots.$$

The binomial series for $(z-2)^{-1}$ converges for $|z| < 2$ and the series for $(z-3)^{-1}$ converges for $|z| < 3$, so as both will converge for $|z| < 2$, this must be the circle of convergence for the required Taylor series. ■

Series obtained by multiplication of series

EXAMPLE 15.11 Find up to the term in z^5 the Maclaurin series expansion of

$$f(z) = \frac{\sin z}{(1+3z^2)}.$$

Solution We will obtain the result by multiplying together an appropriate number of terms of the Maclaurin series expansion of $\sin z$ and the binomial series expansion of $(1+3z^2)^{-1}$. To obtain a result accurate to the term in z^5 we will need to multiply the truncated series

$$\sin z = z - \frac{z^3}{6} + \frac{z^5}{120} + \cdots$$

and the truncated binomial expansion

$$\frac{1}{(1+3z^2)} = 1 - 3z^2 + 9z^4 - \cdots.$$

This gives

$$\begin{aligned}\frac{\sin z}{(1+3z^2)} &= \left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \cdots\right)(1 - 3z^2 + 9z^4 - \cdots)\\ &= z - \frac{19}{6}z^3 + \frac{1141}{120}z^5 - \cdots.\end{aligned}$$

The series for $\sin z$ converges for all z, but the binomial expansion of $(1+3z^2)^{-1}$ only converges for $|z| < 1/\sqrt{3}$, so the required Maclaurin series converges for $|z| < 1/\sqrt{3}$. ■

EXAMPLE 15.12 Find up to the term in z^5 the Maclaurin series expansion of $f(z) = [\log(1-z)]^2$, using the branch of the logarithmic function for which $\log 1 = 2\pi i$.

Solution The principal branch is the function $\mathrm{Log}(1-z)$ for which $\mathrm{Log}\,1 = 0$, and routine differentiation shows the Maclaurin series expansion of $\mathrm{Log}(1-z)$ to be

$$\mathrm{Log}(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \cdots - \frac{z^n}{n} - \cdots.$$

Using the result $e^{2\pi i} = 1$, we can write

$$\begin{aligned}\log(1-z) &= \mathrm{Log}[e^{2\pi i}(1-z)]\\ &= \mathrm{Log}\,e^{2\pi i} + \mathrm{Log}(1-z) = 2\pi i + \mathrm{Log}(1-z),\end{aligned}$$

showing that the appropriate branch of the logarithmic function has the Maclaurin series expansion

$$\log(1-z) = 2\pi i - z - \frac{z^2}{2} - \frac{z^3}{3} - \cdots - \frac{z^n}{n} - \cdots.$$

Multiplying the series $[\log(1-z)]^2$ term by term and collecting all terms up to and including terms in z^5, we obtain

$$\begin{aligned}[\log(1-z)]^2 = -4\pi^2 - 4\pi i z + (1-2\pi i)z^2 + \left(1 - \frac{4\pi i}{3}\right)z^3\\ + \left(\frac{11}{12} - \pi i\right)z^4 + \left(\frac{5}{6} - \frac{4}{5}\pi i\right)z^5 + \cdots.\end{aligned}$$

A careful examination of the coefficients in the series shows that it can be written more systematically as

$$\begin{aligned}[\log(1-z)]^2 = -4\pi^2 + 2\Big[- 2\pi i + (1-2\pi i)\frac{z^2}{2} + \left(1 + \frac{1}{2} - 2\pi i\right)\frac{z^3}{3}\\ + \cdots\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} - 2\pi i\right)\frac{z^n}{n} + \cdots\Big].\end{aligned}$$

The series for $\mathrm{Log}(1-z)$ converges for $|z| < 1$, so the series for $[\log(1-z)]^2$ also converges for $|z| < 1$. ■

Summary

Complex sequences and series have been defined. Tests for the convergence of complex power series were derived that gave rise to the notions of the radius and circle of convergence of the series. These tests are immediate extensions of the corresponding tests for real power series. The complex form of Taylor's theorem was derived, and alternative and often simpler methods for deriving Taylor series were illustrated by example.

EXERCISES 15.1

In Exercises 1 through 4 identify any cluster points that exist, determine whether they belong to the sequence and, where appropriate, find the limit of the sequence. State when a sequence is divergent.

1. (a) $\left\{1 + \frac{(-1)^n}{n}\right\}$. (b) $\left\{[1 + (-1)^n]\left(\frac{2n+1}{n}\right)\right\}$.

(c) $\left\{\frac{5n-1}{2n+6}\right\}$.

2. (a) $\{n^2\}$. (b) $\left\{\frac{n+1}{n} + \frac{(-1)^n}{n^2}\right\}$.

(c) $\left\{n \sin \frac{\pi}{n}\right\}$.

3. (a) $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$. (b) $\left\{\left(\frac{2n^2+1}{n}\right) \tan \frac{\pi}{4n}\right\}$.

(c) $\{1 + \sin n\pi\}$.

4. (a) $\left\{1 + \cos n\pi + \frac{1}{n!}\right\}$. (b) $\left\{\left(\frac{n-1}{n+1}\right)^n\right\}$.

(c) $\left\{\tan\left(\frac{\pi}{2} - \frac{1}{n}\right)\right\}$.

In Exercises 5 through 22 use an appropriate test to determine the nature of the convergence of the series, stating when a series is divergent.

5. $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$.

6. $\sum_{n=1}^{\infty} \frac{2 + (-1)^n}{2^n}$.

7. $\sum_{n=1}^{\infty} \frac{1}{3+n}$.

8. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^3}{5^n}$.

9. $\sum_{n=1}^{\infty} (3\sqrt[n]{n} - 1)^n$.

10. $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$.

11. $\sum_{n=1}^{\infty} (-1)^{n+1} \sin \frac{1}{n^2}$.

12. $\sum_{n=1}^{\infty} \tan^2 \frac{1}{n}$.

13. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

14. $\sum_{n=1}^{\infty} \frac{n(2+i)^n}{2^n}$.

15. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n(2i-1)^n}{3^n}$.

16. $\sum_{n=1}^{\infty} \frac{1}{n(3+i)^n}$.

17. $\sum_{n=1}^{\infty} \frac{1}{n(n-1)}$.

18. $\sum_{n=1}^{\infty} \frac{n - (-1)^n}{3^n}$.

19. $\sum_{n=1}^{\infty} (-1)^{n+1} \left[\frac{n(2-i)+1}{n(3-2i)-3i}\right]^n$.

20. $\sum_{n=1}^{\infty} \left[\frac{2n(4+2i)-1}{n(2+i)+3}\right]^n$.

In Exercises 21 through 34 find the radius of convergence and circle of convergence of the complex power series.

21. $\sum_{n=1}^{\infty} \frac{z^n}{n \cdot 2^n}$.

22. $\sum_{n=1}^{\infty} \frac{z^{2n-1}}{2n-1}$.

23. $\sum_{n=1}^{\infty} \frac{2^{n-1} z^{2n-1}}{(4n-3)^2}$.

24. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}$.

25. $\sum_{n=1}^{\infty} n! z^n$.

26. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n!}$.

27. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)\left(\frac{z}{2}\right)^n$.

28. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(z-5)^n}{n \cdot 3^n}$.

29. $\sum_{n=1}^{\infty} \frac{(z+3)^n}{n^2}$.

30. $\sum_{n=1}^{\infty} \frac{(z-2)^n}{2^n(2n-1)}$.

31. $\sum_{n=1}^{\infty} i^n z^n$.

32. $\sum_{n=1}^{\infty} (1 + ni) z^n$.

33. $\sum_{n=1}^{\infty} \left(\frac{1+2ni}{n+2i}\right)^n z^n$.

34. $\sum_{n=1}^{\infty} \frac{(z-2i)^n}{n \cdot 3^n}$.

In Exercises 35 through 44 use Taylor's theorem to find the first four terms of the expansion of $f(z)$ about the given center.

35. $f(z) = \frac{\sin z}{1 + \sin z}$ with the center $\pi/4$.

36. $f(z) = \cosh(1 + 3z^2)$ with the center 1.

37. $f(z) = \sinh(2 - 3z)$ with the center 1.

38. $f(z) = \operatorname{Log}\left(\frac{4+z}{4-z}\right)$ with the center -1 ($\operatorname{Log} 1 = 0$).

39. $f(z) = \frac{z}{(z+3i)(z-2i)}$ with center i.

40. $f(z) = \cos(2z - i)$ with center i.

41. $f(z) = [\cos z]^2$ with center 0 and $f(0) = 1$.

42. $f(z) = \exp\{z \sin z\}$ with center 0.

43. $f(z) = (z+1)^{1/2}$ with center 0 and $f(0) = (1+i)/\sqrt{2}$.

44. $f(z) = \cos^2(z - i)$ with center $-i$.

In Exercises 45 through 56 use the most appropriate alternative method to find the first four nonvanishing terms in the expansion of $f(z)$ about the given center.

45. $f(z) = \frac{\log(z+1)}{(1+z^2)^{1/2}}$ with the center 0 and $\sqrt{1} = 1$, $\log 1 = 4\pi i$.

46. $f(z) = \frac{z}{(z-3)(z+2)}$ with center -1.

47. $f(z) = \frac{1 - \cos z}{(1-z)^2}$ with center 0.

48. $f(z) = \frac{2z+5}{z^2+z-2}$ with center 2.

49. $f(z) = \dfrac{1+z}{1+2z^2}$ with center 0.

50. $f(z) = \log\left(\dfrac{1+z}{1-z}\right)$ with center 0 and $\log 1 = 2\pi i$.

51. $f(z) = \text{Arctan}\, z$ with center 0 ($\text{Arctan}\, 0 = 0$).

52. $f(z) = [\text{Arctan}\, z]^2$ with center 0 ($\text{Arctan}\, 0 = 0$).

53. $f(z) = \displaystyle\int_0^z \frac{\sin u}{u} du.$

54. $f(z) = (1-z)^{-3}$ with center -1.

55. $f(z) = \dfrac{\sin z}{1-z}$ with center 0.

56. $f(z) = [\cos 2z]^2$ with center $\pi/4$.

15.2 Uniform Convergence

The detailed arguments in this section may be omitted at a first reading, but before doing so, the reader should review the important properties of power series listed in Theorem 15.8.

A power series possesses a special property called *uniform convergence* in any region D of the complex plane where it is convergent. This enables power series to be manipulated as though they were ordinary functions while still retaining the property of uniform convergence.

If $\{u_0(z), u_1(z), u_2(z), \ldots\}$ is an infinite sequence of functions, a series of the form

$$\sum_{n=0}^{\infty} u_n(z) = u_0(z) + u_1(z) + u_2(z) + \cdots \tag{15}$$

is called a **functional series**, and this becomes the *power series*

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots, \tag{16}$$

with its center at z_0 when $u_n(z) = a_n(z-z_0)^n$. As with power series, the ***n*th partial sum** of the functional series (15) is denoted by $s_n(z)$, where

$$s_n(z) = u_0(z) + u_1(z) + u_2(z) + \cdots + u_{n-1}(z). \tag{17}$$

Uniform Convergence

uniform convergence

The functional series (15) is said to **converge uniformly** to the sum $U(z)$ in a region D of the complex plane if for every arbitrary number $\varepsilon > 0$ it is possible to find a number $N = N(\varepsilon)$ that depends on ε, but *not* on z, such that

$$|U(z) - s_n(z)| < \varepsilon \quad \text{for all } n > N \quad \text{and all } z \text{ in } D. \tag{18}$$

It follows from this definition that the power series (16) will be uniformly convergent in D if it can be shown that

$$\left|\sum_{k=n}^{\infty} a_k(z-z_0)^k\right| < \varepsilon \quad \text{for all } n > N \quad \text{and all } z \text{ in } D. \tag{19}$$

A comparison of the definitions of uniform convergence and convergence shows that whereas for convergence the number N depends on ε and the value of z, in the case of uniform convergence the number N depends only on ε, and *not* on z. It is because of the independence of the convergence on the value of z in D that the term *uniform* is used to describe this powerful form of convergence.

In practical terms, if a power series converges uniformly to $f(z)$ in a circle of convergence D, and it is known that when $n = N$ the Nth partial sum $s_N(z)$ at a point z_0 in D approximates $f(z_0)$ in such a way that

$$|f(z_0) - s_N(z_0)| < \varepsilon$$

for some known small number $\varepsilon > 0$, then $s_N(z)$ will approximate $f(z)$ with the *same* accuracy for *all* points z in D. This is *not* the case for series that are not uniformly convergent, because in that case the number of terms needed in the partial sum to maintain the accuracy will depend on the value of z.

The following theorem, called the **Weierstrass *M*-test**, provides the simplest test for uniform convergence.

THEOREM 15.10

the simplest test for uniform convergence

Weierstrass *M*-test Let the functional series $\sum_{n=0}^{\infty} u_n(z)$ be such that for each n, $|u_n(z)| < M_n$ for all z in a domain D. Then if the series of positive constants $\sum_{n=0}^{\infty} M_n$ converges, the series $\sum_{n=0}^{\infty} u_n(z)$ is uniformly convergent in D.

Proof Let $s_n(z)$ and $s_{n+p}(z)$ be the nth and $(n+p)$th partial sums of the series, with $p > 0$ any positive integer. Then

$$\begin{aligned} |s_{n+p}(z) - s_n(z)| &= |u_n(z) + u_{n+1}(z) + \cdots + u_{n+p}(z)| \\ &\leq |u_n(z)| + |u_{n+1}(z)| + \cdots + |u_{n+p}(z)| = \sum_{k=n}^{n+p} M_k, \end{aligned}$$

where repeated use has been made of the triangle inequality.

By hypothesis the series $\sum_{n=0}^{\infty} M_n$ is convergent, so it follows from the Cauchy convergence principle that the sum $\sum_{k=n}^{n+p} M_k$ can be made arbitrarily small by making n sufficiently large, so a function $U(z)$ exists such that $U(z) = \lim_{n\to\infty} s_n(z)$. This has established that the conditions of the theorem ensure that the functional series is *convergent*.

To show that the convergence is *uniform* it is only necessary to notice that for any $\varepsilon > 0$, the convergence of $\sum_{n=0}^{\infty} M_n$ means that a positive integer $N(\varepsilon)$ can be found such that if $n \geq N(\varepsilon)$, then $\sum_{k=n}^{\infty} M_k < \varepsilon$. So, for $n \geq N(\varepsilon)$ and all z in D,

$$|U(z) - s_n(z)| = \left|\sum_{k=n}^{\infty} u_k(z)\right| \leq \sum_{k=n}^{\infty} M_k < \varepsilon,$$

and the theorem is proved. ■

EXAMPLE 15.13

Prove that the power series $\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots + z^n + \cdots$ is uniformly convergent inside the unit circle $|z| = 1$.

Solution Let z^* be a point inside the unit circle $|z| = 1$ and write $|z^*| = r$, so $r < 1$ and $|(z^*)^n| = r^n < 1$. Setting $M_n = r^n$ in the Weierstrass M-test, we obtain

$$\sum_{n=0}^{\infty} M_n < \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \quad \text{with } r < 1.$$

As the conditions of the theorem are satisfied, the series is uniformly convergent everywhere inside the unit circle $|z| = 1$.

This result is not unexpected, because $\sum_{n=0}^{\infty} z^n$ is the Maclaurin series expansion of $1/(1-z)$, and this is an analytic function inside the unit circle. ■

From now on attention will be confined to power series, and the next theorem generalizes the result of the last example by proving that every power series converges uniformly inside its circle of convergence.

THEOREM 15.11

a power series converges uniformly inside its circle of convergence

Uniform convergence of power series A power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ with a radius of convergence $R > 0$ converges uniformly inside and on every circle $|z - z_0| = r$, where $r < R$.

Proof The proof of the theorem makes use of the Weierstrass M-test. From the definition of the radius of convergence of a series it follows that the series $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ is absolutely convergent for $|z - z_0| < r$, so for any $z = \zeta$ on the circle $|z - z_0| = r$ the series

$$\sum_{n=0}^{\infty} |a_n(\zeta - z_0)^n| = \sum_{n=0}^{\infty} |a_n| r^n$$

must also be convergent. Hence, for all z inside and on the circle $|z - z_0| = r$, the following inequality must hold:

$$|a_n(z-z_0)^n| \leq |a_n| r^n.$$

The statement of the theorem now follows from this result and the convergence of the series $\sum_{n=0}^{\infty} |a_n| r^n$ if we apply the Weierstrass M-test to the power series with $M_n = |a_n| r^n$. ■

The result of Example 15.13 is a special case of this theorem. An examination of the series $\sum_{n=0}^{\infty} z^n$ shows that its radius of convergence is $R = 1$, so as the series is a power series expansion with the origin as center it is uniformly convergent inside the circle $|z| = 1$, as was shown directly in the example.

It is useful to use Theorem 15.11 to reformulate the results of Theorem 15.8 concerning the differentiation and integration of power series.

THEOREM 15.12

a power series can be differentiated and integrated inside its circle of convergence

Differentiation and integration of power series Let a power series with the sum $f(z)$ have a circle of convergence $|z - z_0| = R$, where $R > 0$. Then the series possesses the following properties:

(i) The power series converges uniformly to $f(z)$ inside the circle of convergence.

(ii) The power series obtained by term-by-term differentiation of the power series for $f(z)$ converges uniformly to $f'(z)$ and has the same circle of convergence as

$f(z)$, so if

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \quad \text{then} \quad f'(z) = \sum_{n=1}^{\infty} na_n(z - z_0)^{n-1}.$$

(iii) The power series obtained by term-by-term integration of the power series for $f(z)$ along any path Γ inside the circle of convergence converges uniformly to the integral of $f(z)$ along Γ, so if

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \quad \text{then} \quad \int_\Gamma f(z)dz = \sum_{n=0}^{\infty} a_n \int_\Gamma (z - z_0)^n dz.$$ ■

EXAMPLE 15.14 Use the Maclaurin series for $1/(1-2z)$ to find the Maclaurin series for $1/(1-2z)^2$, and confirm that both series have the same circle of convergence.

Solution The Maclaurin series expansion is obtained most easily by writing the function in the form $(1-2z)^{-1}$ and expanding it by the binomial theorem, when we obtain

$$\frac{1}{1-2z} = (1-2z)^{-1} = 1 + 2z + 2^2z^2 + \cdots = \sum_{n=0}^{\infty} 2^n z^n.$$

This binomial series is convergent for $|z| < 1/2$, so this is the circle of convergence for the function. By Theorem 15.12 (ii) this series can be differentiated term by term inside its circle of convergence, so as

$$\frac{d}{dz}\left(\frac{1}{1-2z}\right) = \frac{2}{(1-2z)^2} \quad \text{and} \quad \frac{d}{dz}\sum_{n=0}^{\infty}[2^n z^n] = 2 + 2\cdot 2^2 z + 3\cdot 2^3 z^2 + \cdots$$

$$= \sum_{n=0}^{\infty}(n+1)2^{n+1}z^n,$$

equating these results and cancelling a factor 2 gives the desired expansion,

$$\frac{1}{(1-2z)^2} = \sum_{n=0}^{\infty}(n+1)2^n z^n = 1 + 4z + 12z^2 + \cdots.$$

It is easily verified that this power series has a radius of convergence $R = \frac{1}{2}$, so the differentiated series is also uniformly convergent for $|z| < \frac{1}{2}$. ■

EXAMPLE 15.15 By integrating the Maclaurin series for $\sin\zeta$ along a suitable path, find the Maclaurin series for $\cos z$.

Solution The Maclaurin series for $\sin\zeta$ is

$$\sin\zeta = \sum_{n=0}^{\infty}(-1)^n\frac{\zeta^{2n+1}}{(2n+1)!} = \zeta - \frac{\zeta^3}{3!} + \frac{\zeta^5}{5!} - \cdots.$$

As this power series converges for all finite ζ, it follows from Theorem 15.12 (iii) that term-by-term integration is permitted along any path in the complex plane, so integrating from the origin to an arbitrary point z gives

$$\int_0^z \sin\zeta\, d\zeta = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\int_0^z \zeta^{2n+1}d\zeta.$$

After the integrations are performed this becomes

$$1 - \cos z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+2}}{(2n+2)!},$$

and a rearrangement of terms leads to the expected result

$$\cos z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!},$$

where the series on the right also converges for all finite z. ■

EXAMPLE 15.16 By integrating the Maclaurin series for $1/(1+\zeta)$ along a suitable path in its circle of convergence, show that

$$\text{Log}(1+z) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{n} \quad \text{for } |z| < 1.$$

Solution The Maclaurin series expansion of $1/(1+\zeta)$ is most easily found by means of the binomial theorem, so we can write

$$\frac{1}{1+\zeta} = 1 - \zeta + \zeta^2 - \zeta^3 + \cdots = \sum_{n=0}^{\infty}(-1)^n \zeta^n.$$

This power series has a radius of convergence $R = 1$ and so is uniformly convergent inside the circle of convergence $|\zeta| = 1$. By Theorem 15.12 (iii), this series can be integrated term by term along any path Γ inside this circle, so

$$\int_{\Gamma} \frac{1}{1+\zeta} d\zeta = \sum_{n=0}^{\infty}(-1)^n \int_{\Gamma} \zeta^n d\zeta.$$

To obtain $\text{Log}(1+z)$ we choose Γ to be the straight line path joining the origin to a point $\zeta = z$ inside the circle of convergence, and take the *principal* branch of the logarithmic function as an antiderivative of the integral on the left. As a result, on the left we obtain

$$\int_0^z \frac{1}{1+\zeta} d\zeta = \text{Log}(1+z), \quad \text{where} \quad \text{Log}(1+\zeta) = \ln|1+\zeta| + i\theta,$$

where θ is the argument of $\text{Log}(1+\zeta)$, with $-\pi < \theta \leq \pi$. Integration of the expression on the right leads to the result

$$\sum_{n=0}^{\infty}(-1)^n \int_0^z \zeta^n d\zeta = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{n},$$

so equating these expresions gives

$$\text{Log}(1+z) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{n}.$$

Care must always be exercised when working with logarithmic functions because they are multivalued. The principal branch of $\text{Log}(1+z)$ used here is analytic throughout the complex plane, with the exception of the branch cut made along the negative real axis from $-\infty$ to the point $z = -1$. However, the series representation of $\text{Log}(1+z)$ is only valid inside the circle $|z| = 1$ where, like the function

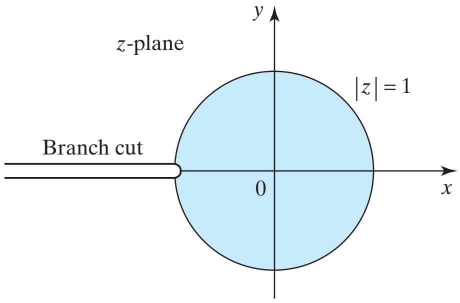


FIGURE 15.3 The circle of convergence for the series representation of $\text{Log}(1 + z)$ and the branch cut for the function $\text{Log}(1 + z)$.

$f(z) = 1/(1 + z)$, it is analytic. Figure 15.3 shows the circle of convergence for the series expansion of $\text{Log}(1 + z)$, and the branch cut used in the definition of the function $\text{Log}(1 + z)$. ■

Summary

The concept of uniform convergence was defined and related to power series, and the simple Weierstrass *M*-test for uniform convergence was given. The importance of the uniform convergence of a power series was shown to be that it retains its uniform convergence property when it is either differentiated or integrated inside its circle of convergence, thereby allowing it to be manipulated like an ordinary function.

15.3 Laurent Series and the Classification of Singularities

We have seen how a function $f(z)$ that is analytic at a point z_0 can be expanded in a neighborhood of z_0 as a Taylor series with z_0 as its center. Although Taylor series expansions are sufficient for many purposes, the requirement that $f(z)$ must be analytic at z_0 means some other form of expansion must be used when an expansion is required about a point where $f(z)$ is not analytic. The development of a more general form of expansion that overcomes this difficulty leads to what is called a *Laurent series* expansion of a function. Arising from the study of Laurent series comes the need to classify the nature of points where a function is not analytic.

Points where a function $f(z)$ is analytic are called **regular points** of the function, and a point z_0 where $f(z)$ is analytic in every neighborhood of z_0, but not at z_0 itself, is called a **singular point** of the function. For example, the function $f(z) = 1/z$ is analytic for all finite z apart from the point $z = 0$ where its derivative is not defined, so the origin is a singular point of $f(z) = 1/z$.

regular and singular points and Laurent series

A **Laurent series** $L(z)$ is a series of the form

$$L(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{20}$$

that contains both positive and negative powers of $(z - z_0)$. It is customary to

represent a Laurent series $L(z)$ as the sum of two series by setting $L(z) = L_1(z) + L_2(z)$ where

$$L_1(z) = \sum_{n=-\infty}^{-1} a_n(z - z_0)^n \quad \text{and} \quad L_2(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n. \tag{21}$$

The series $L_1(z)$ containing only negative powers of $(z - z_0)$ is called the **principal part** of the Laurent series, and the series $L_2(z)$ containing only positive powers of $(z - z_0)$ is called its **regular part**. A Laurent series is said to **converge** in a domain D when both of the series $L_1(z)$ and $L_2(z)$ are convergent in D. In general, a Laurent series converges in an annulus

$$r < |z - z_0| < R, \quad \text{where} \quad 0 < r < R.$$

A simple example of a Laurent series is obtained by considering the function $(\cos z)/z$ and expanding $\cos z$ as a Maclaurin series to arrive at the representation

$$\frac{\cos z}{z} = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n-1}}{(2n)!} = \frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} - \cdots.$$

The principal part of this Laurent series is the single term $L_1(z) = 1/z$, and its regular part is the power series

$$L_2(z) = \sum_{n=1}^{\infty}(-1)^n \frac{z^{2n-1}}{(2n)!} = -\frac{z}{2!} + \frac{z^3}{4!} - \frac{z^5}{6!} + \cdots.$$

In this case the principal part of the expansion is finite (converges) for all $z \neq 0$, and the regular part converges for all z, so the annulus in which this Laurent series converges becomes the complex plane from which has been deleted the single point at the origin.

The next theorem shows how a function that is analytic in an annulus with its center at the point z_0 can be expanded inside the annulus as a unique Laurent series. This theorem also provides an explicit general formula for the Laurent coefficients. The examples that follow the theorem show how simple algebraic arguments often provide easier ways of finding the Laurent coefficients than using the general formula.

THEOREM 15.13

the Laurent expansion theorem

Laurent's theorem A function $f(z)$ that is analytic in the annulus D given by $R_1 < |z - z_0| < R_2$ can be expanded in D as a unique Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \quad \text{with } n = 0, \pm 1, \pm 2, \ldots,$$

and Γ is any positively oriented circle in D given by $|\zeta - z_0| = \rho$, with $R_1 < \rho < R_2$.

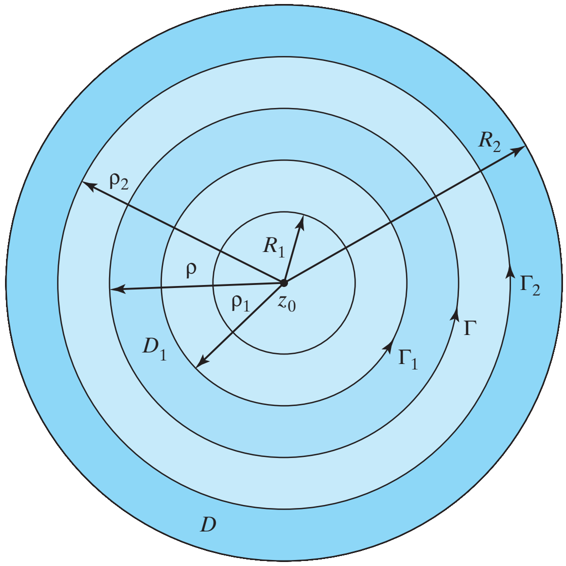


FIGURE 15.4 The annulus D determined by $R_1 < |z - z_0| < R_2$.

Proof Let the annulus D be the one shown in Fig. 15.4 with its center at z_0, its inner boundary a circle of radius R_1, and its outer boundary a circle of radius R_2. The positively oriented circles Γ_1 and Γ_2 with the respective radii ρ_1 and ρ_2 bound the annulus D_1 contained in D, where the positively oriented circle Γ inside D_1 has radius ρ.

If z is a fixed point inside D_1, then by the extended Cauchy–Goursat theorem we can write

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_2} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i}\int_{\widetilde{\Gamma}_1} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where $\widetilde{\Gamma}_1$ denotes integration around Γ_1 in the negative (clockwise) sense.

In the integrand of the first term ζ lies on Γ_2, so we expand $1/(\zeta - z)$ as the power series in $z - \zeta$:

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0)\left(1 - \frac{z - z_0}{\zeta - z_0}\right)} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}.$$

This is a geometric series, and because ζ lies on Γ_2 we have $|\zeta - z_0| = \rho_2$, showing that

$$\left|\frac{z - z_0}{\zeta - z_0}\right| = \frac{|z - z_0|}{\rho_2} < 1.$$

Applying the Weierstrass M-test shows that the series expansion of $1/(\zeta - z)$ is uniformly convergent. As uniform convergence allows term-by-term integration of the series, substituting the series expansion in the integral gives

$$\frac{1}{2\pi i}\int_{\Gamma_2} \frac{f(\zeta)}{\zeta - z} d\zeta = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi i}\int_{\Gamma_2} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta.$$

A similar argument can be used to express the integrand in the second integral as

$$\frac{1}{\zeta - z} = \frac{1}{z - z_0 - (\zeta - z_0)} = \frac{-1}{(z - z_0)\left(1 - \frac{\zeta - z_0}{z - z_0}\right)} = -\sum_{k=0}^{\infty} \frac{(\zeta - z_0)^k}{(z - z_0)^{k+1}},$$

where, as ζ now lies on Γ_1,

$$\left|\frac{\zeta - z_0}{z - z_0}\right| = \frac{\rho_1}{|z - z_0|} < 1.$$

The Weierstrass M-test shows this series is also uniformly convergent. Substituting the series in the second integral and integrating term by term gives

$$\frac{1}{2\pi i}\int_{\tilde{\Gamma}_1} \frac{f(\zeta)}{\zeta - z} d\zeta = -\frac{1}{2\pi i}\int_{\Gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i}\sum_{k=0}^{\infty}\int_{\Gamma_1} \frac{f(\zeta)(\zeta - z_0)^k}{(z - z_0)^{k+1}} d\zeta,$$

where a negative sign has been introduced to compensate for the change from contour $\tilde{\Gamma}_1$ where integration is in the clockwise sense, to contour Γ_1 where the integration is counterclockwise.

When $k + 1$ is replaced by $-n$ the summation becomes

$$\frac{1}{2\pi i}\int_{\Gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta = \sum_{n=-1}^{-\infty} a_n (z - z_0)^n,$$

with

$$a_n = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta.$$

Combining the two integrals, and recognizing that the positively oriented circles Γ_1 and Γ_2 bounding D_1 may both be deformed into any positively oriented contour Γ that lies in D_1 with z_0 in its interior, shows that the Laurent series coefficients a_n are *all* given by the single formula

$$a_n = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \quad \text{with } n = 0, \pm 1, \pm 2, \ldots.$$

Finally, as the fixed point z was *any* point inside the annulus D_1 that is itself contained in the annulus D, the first part of the theorem has been proved.

The uniqueness of a Laurent series expansion in a given annulus can be established as follows. Suppose, if possible, that $f(z)$ can be represented in the same annulus by the two different Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \sum_{n=-\infty}^{\infty} b_n (z - z_0)^n.$$

Forming the product of these series with $(z - z_0)^{-m-1}$, where m is a fixed integer, leads to the result

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^{n-m-1} = \sum_{n=-\infty}^{\infty} b_n (z - z_0)^{n-m-1}.$$

Each of these series converges on the contour Γ inside D_1, so using the results

$$\int_\Gamma (z - z_0)^k dz = \begin{cases} 0, & k \neq -1 \\ 2\pi i, & k = -1 \end{cases} \quad (k \text{ a positive or negative integer})$$

shows that $a_k = b_k$ for each k, so the uniqueness of the Laurent series is proved. ■

PIERRE-ALPHONSE LAURENT (1813–1854)
A French mathematician whose major contribution to complex analysis, published in 1843, was the fact that when a function is discontinuous at a single point, the Taylor series expansion of the function must be replaced by an expansion involving both increasing and decreasing powers of the variable involved. This result is the one now known as Laurent's theorem.

The uniqueness of a Laurent series expansion of an analytic function $f(z)$ in a given annulus means that any method used to generate the expansion in the annulus will produce the same series. This result can be used to considerable advantage, because instead of using the general formula given in Theorem 15.13, it frequently proves to be easier to find the coefficients of the series by using a simple algebraic approach.

If an analytic function $f(z)$ that is expanded about the point z_0 has singular points at $a_1, a_2, \ldots, a_n$, then the loss of differentiability at these points means that the radius R_2 of the outer circle of the annulus in which the expansion is valid cannot exceed the distance from z_0 to the *nearest* singular point, so that

$$R_2 = \min\{|z_0 - a_1|, |z_0 - a_2|, \ldots, |z_0 - a_n|\}.$$

how algebraic arguments can often simplify the task of finding a Laurent series

The expansion will, of course, be analytic everywhere on the outer boundary of the annulus of convergence except for any point where there is a singularity.

The next example illustrates the use of algebraic arguments to develop Laurent series, and also how the location of singularities relative to the point about which the expansion is carried out determines the outer radius of the annulus of convergence.

EXAMPLE 15.17

Find the Laurent series expansion of

$$f(z) = \frac{1}{6 - z - z^2}$$

in (a) the domain D_1 determined by $|z| < 2$, (b) the domain D_2 determined by $2 < |z| < 3$, and (c) the domain D_3 determined by $|z| > 3$.

Solution Factoring the denominator gives

$$f(z) = \frac{1}{(2 - z)(z + 3)},$$

so the function has singular points at $z = 2$ and $z = -3$, but is analytic elsewhere. As these points occur on the boundaries of the domains D_1, D_2, and D_3, the function will be analytic inside each of these domains. Consequently, $f(z)$ will have a unique though different Laurent series expansion in each of the three domains. The required expansions will now be obtained by using simple algebraic arguments that

start from the partial fraction decomposition

$$f(z) = \frac{1}{5}\left(\frac{1}{2-z} + \frac{1}{z+3}\right).$$

If $|z| < 2$, by using the binomial theorem we can write

$$\frac{1}{2-z} = \frac{1}{2\left(1-\frac{z}{2}\right)} = \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} = \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}.$$

If $|z| > 2$, it follows in similar fashion that we can write

$$\frac{1}{2-z} = \frac{1}{z\left(\frac{2}{z}-1\right)} = -\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} = -\sum_{n=1}^{\infty}\frac{2^{n-1}}{z^n}.$$

If $|z| < 3$, we can write

$$\frac{1}{z+3} = \frac{1}{3\left(1+\frac{z}{3}\right)} = \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1} = \sum_{n=0}^{\infty}(-1)^n\frac{z^n}{3^{n+1}}.$$

Finally, if $|z| > 3$, we have

$$\frac{1}{z+3} = \frac{1}{z\left(1+\frac{3}{z}\right)} = \frac{1}{z}\left(1+\frac{3}{z}\right)^{-1} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^{n-1}}{z^n}.$$

These results can now be combined with the partial fraction decomposition to obtain the Laurent series expansions in each of the three domains.

(a) In D_1 where $|z| < 2$ we have from the first and third of the preceding expansions that

$$f(z) = \sum_{n=0}^{\infty}\frac{1}{5}\left[\frac{1}{2^{n+1}} + \frac{(-1)^n}{3^{n+1}}\right]z^n.$$

This expansion contains no principal part, and because $f(z)$ is analytic in D_1 we see that in this domain the Laurent series has degenerated into a Taylor series expansion about the origin that is, of course, just the Maclaurin series expansion of $f(z)$ in D_1.

(b) In D_2 where $2 < |z| < 3$, we have from the second and third of the preceding expansions that

$$f(z) = \sum_{n=1}^{\infty}\left(-\frac{2^{n-1}}{5}\right)\frac{1}{z^n} + \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{5\cdot 3^{n+1}}\right)z^n.$$

Here the first summation represents the principal part and the second summation the regular part of the Laurent series expansion in the domain.

(c) In D_3 where $|z| > 3$, we have from the second and fourth of the preceding expansions that

$$f(z) = \sum_{n=1}^{\infty}\frac{1}{5}[-2^{n-1} + (-1)^{n-1}3^{n-1}]\frac{1}{z^n}.$$

This shows that in D_3 the Laurent series expansion has only a principal part.

Although expansions (a) and (b) are different in form, each is analytic on the circle $|z| = 2$, with the exception of the point $z = 2$ where a singularity occurs. Thus,

representations (a) and (b) give different, but equivalent, representations of $f(z)$ on the circle $|z| = 2$ away from the single point $z = 2$. A similar situation occurs with representations (b) and (c) on the circle $|z| = 3$ away from the single point $z = -3$ where the other singularity is located. ■

EXAMPLE 15.18 Expand $f(z) = \exp\left(z + \frac{1}{z}\right)$ as a Laurent series about the origin.

Solution The function $f(z)$ is analytic everywhere except at the origin, which is a singular point. Consequently, when $f(z)$ is expanded about the origin its Laurent series will converge throughout the complex plane with the exception of the single point $z = 0$, and the series will be of the form

$$\exp\left(z + \frac{1}{z}\right) = \sum_{n=1}^{\infty} a_{-n}\frac{1}{z^n} + \sum_{n=0}^{\infty} a_n z^n, \quad \text{for } |z| > 0.$$

To determine the coefficients $a_{\pm n}$, we write the function as $f(z) = (\exp z)(\exp \frac{1}{z})$ and then express this as the product of the two series

$$(\exp z)\left(\exp\frac{1}{z}\right) = \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \cdots\right)$$
$$\times\left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \frac{1}{4!z^4} + \frac{1}{5!z^5} + \cdots\right).$$

The coefficient a_0 is simply the constant term in this product, so identifying this as the sum of products of the form $\frac{z^k}{k!} \cdot \frac{1}{z^k k!}$, we find that

$$a_0 = 1 + 1 + \frac{1}{(2!)^2} + \frac{1}{(3!)^2} + \frac{1}{(4!)^2} + \cdots = \sum_{k=0}^{\infty} \frac{1}{(k!)^2}.$$

Further examination of the product of the two series shows that the coefficients a_n and a_{-n} are equal, so we need only determine a_n. The coefficient a_1 in the Laurent series expansion about the origin is the coefficient of z in the preceding product, so identifying this as the sum of the products $\frac{z^{k+1}}{(k+1)!}\frac{1}{z^k k!}$ gives

$$a_1 = a_{-1} = 1 + \frac{1}{2!} + \frac{1}{2!\cdot 3!} + \frac{1}{3!\cdot 4!} + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!}.$$

If we proceed in this manner, it is not difficult to see that

$$a_n = a_{-n} = \sum_{k=0}^{\infty} \frac{1}{k!(n+k)!}.$$

Substituting these values for a_0 and $a_{\pm n}$ into

$$\exp\left(z + \frac{1}{z}\right) = \sum_{n=1}^{\infty} a_{-n}\frac{1}{z^n} + \sum_{n=0}^{\infty} a_n z^n$$

gives the required Laurent series expansion that is convergent for $|z| > 0$. ■

EXAMPLE 15.19 Find (a) the Laurent series expansion of $f(z) = 1/(z^2 + 1)^2$ in the largest possible circle about the point $z = i$, and (b) the expansion about the origin for $|z| > 1$.

Solution

(a) Writing $f(z)$ as

$$f(z) = \frac{1}{(z-i)^2(z+i)^2}$$

shows that the function has singularities only at $z = i$ and $z = -i$. When the function is expanded in a Laurent series about $z = i$, the radius R of the outer boundary of the largest annulus of convergence must equal the distance between $z = i$ and the singularity at $z = -i$ closest to $z = i$. As $|i - (-i)| = 2$ we see that $R = 2$, so as the point $z = i$ must be excluded from the annulus of convergence centered on $z = i$, the function $f(z)$ will be analytic in the punctured disc $0 < |z - i| < 2$, where the expansion will be in terms of powers of $z - i$.

Simplifying $f(z)$ by using partial fractions gives

$$f(z) = \frac{1}{(z^2+1)^2} = -\frac{i}{4}\frac{1}{z-i} - \frac{1}{4}\frac{1}{(z-i)^2} + \frac{i}{4}\frac{1}{z+i} - \frac{1}{4}\frac{1}{(z+i)^2}.$$

The first two terms are already expressed in terms of powers of $z - i$, so it remains to express the last two terms in this form.

The third term on the right can be written as

$$\frac{i}{4}\frac{1}{z+i} = \frac{i}{4}\frac{1}{(z-i)+2i},$$

so as $|z - i| < |2i|$ the binomial theorem can be used to expand this expression as

$$\begin{aligned}\frac{i}{4}\frac{1}{z+i} &= \frac{i}{4}\frac{1}{2i}\frac{1}{\left[1+\frac{z-i}{2i}\right]} = \frac{1}{8}\left[1 - i\left(\frac{z-i}{2}\right)\right]^{-1}\\ &= \frac{1}{8}\sum_{n=0}^{\infty}\frac{i^n(z-i)^n}{2^n} = \sum_{n=0}^{\infty}\frac{i^n(z-i)^n}{2^{n+3}}.\end{aligned}$$

The fourth term can be written in a similar form by writing

$$\begin{aligned}-\frac{1}{4}\frac{1}{(z+i)^2} &= -\frac{1}{4}\frac{1}{[(z-i)+2i]^2} = -\frac{1}{4}\frac{1}{(2i)^2}\frac{1}{\left[1+\frac{z-i}{2i}\right]^2} = \frac{1}{16}\left[1 - i\left(\frac{z-i}{2}\right)\right]^{-2}\\ &= \frac{1}{16}\sum_{n=1}^{\infty}\frac{ni^{n-1}(z-i)^{n-1}}{2^{n-1}}.\end{aligned}$$

The coefficients of the Laurent series expansion will be simplified if the last two results are combined. To accomplish this, we change the summation index in the last expansion to make it start from zero. This is accomplished by setting $n - 1 = m$ when we can write

$$\frac{1}{16}\sum_{n=1}^{\infty}\frac{ni^{n-1}(z-i)^{n-1}}{2^{n-1}} = \sum_{m=0}^{\infty}\frac{(1+m)i^m(z-i)^m}{2^{m+4}}.$$

As the choice of symbol for a summation index does not affect the summation, we now replace m by n to obtain the equivalent result

$$-\frac{1}{4}\frac{1}{(z+i)^2} = \sum_{n=0}^{\infty}\frac{(1+n)i^n(z-i)^n}{2^{n+4}}.$$

As a result, the last two terms of the partial fraction decomposition become

$$\frac{i}{4}\frac{1}{z+i} - \frac{1}{4}\frac{1}{(z+i)^2} = \sum_{n=0}^{\infty}\frac{i^n(z-i)^n}{2^{n+3}} + \sum_{n=0}^{\infty}\frac{(1+n)i^n(z-i)^n}{2^{n+4}}$$

$$= \sum_{n=0}^{\infty}\frac{(n+3)i^n(z-i)^n}{2^{n+4}},$$

from which the complete Laurent series expansion for $0 < |z-i| < 2$ is seen to be

$$\frac{1}{(z^2+1)^2} = -\frac{i}{4}\frac{1}{z-i} - \frac{1}{4}\frac{1}{(z-i)^2} + \sum_{n=0}^{\infty}\frac{(n+3)i^n(z-i)^n}{2^{n+4}}.$$

(b) The singularities of $f(z)$ occur on the unit circle $|z| = 1$, so outside this circle the function will be analytic. As $|1/z| < 1$ in the required domain, the binomial theorem can be used to expand the function when written in the form

$$\frac{1}{(z^2+1)^2} = \frac{1}{z^4}\frac{1}{\left(1+\frac{1}{z^2}\right)^2} = \frac{1}{z^4}\left(1+\frac{1}{z^2}\right)^{-2},$$

from which it follows that

$$\frac{1}{(z^2+1)^2} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{z^{2n+2}} \quad \text{for } |z| > 1.$$

When $|z|$ is large the operations leading to a Laurent series are sometimes difficult to perform directly. In such circumstances the substitution $z = 1/u$ is made where $|u|$ is small, corresponding to $|z|$ large, and after the expansion has been developed in terms of u, the result is then transformed back to the original variable z. This approach is illustrated in the next example.

EXAMPLE 15.20 Find the Laurent series expansion of $f(z) = \mathrm{Log}(\frac{z-1}{z-2})$ for large $|z|$.

Solution Substituting $z = 1/u$ in $f(z)$ gives

$$f(z) = \mathrm{Log}\left(\frac{z-1}{z-2}\right) = \mathrm{Log}\left(\left(1-\frac{1}{z}\right)\Big/\left(1-\frac{2}{z}\right)\right) = \mathrm{Log}\left(\frac{1-u}{1-2u}\right)$$

$$= \mathrm{Log}(1-u) - \mathrm{Log}(1-2u).$$

Replacing the logarithms in this last expression by their Maclaurin series expansions that will *both* be valid provided $|u| < \frac{1}{2}$ gives

$$f(u) = -\left(u + \frac{1}{2}u^2 + \frac{1}{3}u^3 + \cdots \frac{1}{n}u^n + \cdots\right) + \left(2u + 2u^2 + \frac{8}{3}u^3 + \cdots + \frac{2^n}{n}u^n + \cdots\right)$$

$$= u + \frac{3}{2}u^2 + \frac{7}{3}u^3 + \cdots + \left(\frac{2^n-1}{n}\right)u^n + \cdots, \quad \text{for } |u| < \frac{1}{2}.$$

Finally, transforming back to the variable z, and noticing that $|u| < 2$ corresponds to $|z| > 2$, we arrive at the required Laurent series expansion for large $|z|$:

$$\mathrm{Log}\left(\frac{z-1}{z-2}\right) = \sum_{n=1}^{\infty}\frac{2^n-1}{nz^n}, \quad \text{for } |z| > 2.$$

The expansion of functions as Laurent series makes it necessary to classify the different types of singularity that arise. The relevance of this classification, and the importance of the coefficients of a Laurent series, will become clear later once the evaluation of integrals by means of contour integration has been developed.

isolated singularities, removable singularities, poles, and essential singularities

A point z_0 is called an **isolated singularity** of a function $f(z)$ if $f(z)$ has a singularity at z_0, but is single valued and analytic in the annulus (punctured disc) $0 < |z - z_0| < R$.

Singularities are easily identified when a function is a quotient of analytic functions $f(z) = g(z)/h(z)$, because they occur at any zero z^* of $h(z)$ where the numerator $g(z^*) \neq 0$, and also at any infinity of $g(z)$ where $h(z)$ remains finite.

For example, the function $f(z) = (z+3)/(z^2+4)$ has singularities at the zeros $z = \pm 2i$ of the denominator $z^2 + 4$, because the numerator $z + 3$ does not vanish at either of these points. However, the function $f(z) = (\tan z)/z^2$ has a singularity at $z = 0$ due to a zero of the denominator, because although $\tan z = z + z^3/3 + \ldots$, the function $f(z) = (\tan z)/z^2 = (1/z) + z/3 + \ldots$. So $f(z)$ has a singularity at the origin also and also at $z = (2n+1)\pi/2$ for $n = 0, \pm 1, \pm 2, \ldots$, because of infinities of the numerator.

Consideration of the general form of the Laurent series expansion given in (15) allows three distinct cases to be identified, namely:

1. The Laurent series for $f(z)$ contains no negative powers of $(z - z_0)$.
2. The Laurent series for $f(z)$ only contains a finite number of terms involving negative powers of $(z - z_0)$, up to and including the term in $(z - z_0)^{-r}$.
3. The Laurent series for $f(z)$ contains infinitely many terms involving negative powers of $(z - z_0)$.

Case 1. Functions $f(z)$ with this property are said to have a **removable singularity** at z_0 because, irrespective of how $f(z)$ is defined at z_0 (and even if it is not defined), the Laurent series converges to the value a_0 when $z = z_0$. Consequently, by defining $f(z_0) = a_0$ the singularity (discontinuity) at z_0 is *removed*. In working with functions with removable singularities, it is always assumed that they have been removed.

Case 2. Functions $f(z)$ with this property have a principal part of the Laurent series of the form

$$\frac{a_{-1}}{(z-z_0)} + \frac{a_{-2}}{(z-z_0)^2} + \cdots + \frac{a_{-r+1}}{(z-z_0)^{r-1}} + \frac{a_{-r}}{(z-z_0)^r},$$

where some or all of the coefficients $a_{-1}, a_{-2}, \ldots, a_{-r+1}$ may be zero, but $a_{-r} \neq 0$. This type of singularity is called a **pole of order *r*** of the function $f(z)$ located at z_0, or sometimes a pole of **multiplicity *r*** located at z_0. A pole of order 1 is called a **simple pole**.

Although no further use will be made of the term, for the sake of completeness we mention that the quotient of two analytic functions is called a **meromorphic function**. Thus, a meromorphic function is analytic throughout a domain apart from points where poles arise due to a zero of the denominator where the numerator is nonvanishing.

Case 3. Functions $f(z)$ with this property are said to have an **essential singularity** located at the point z_0.

In what follows our concern will only be with Cases 1 and 2, because of the extremely erratic behavior of functions in a neighborhood of an essential singularity.

EXAMPLE 15.21 Identify the singularities of the functions

(a) $f(z)=\dfrac{\cosh z-1}{z^2}$, (b) $f(z)=\dfrac{2z^2+13z+3}{z^3+3z^2-4}$, (c) $f(z)=\dfrac{\sinh z}{z^5}$,

(d) $f(z)=z\exp(1/z)$.

Solution

(a) $f(z)$ is analytic everywhere apart from $z=0$ where it is indeterminate. To examine the behavior of $f(z)$ at the origin we replace $\cosh z$ by its Maclaurin series, leading to the result

$$\frac{\cosh z-1}{z^2}=\frac{\left(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots\right)-1}{z^2}.$$

Cancelling the 1 and dividing by z^2 gives

$$\frac{\cosh z-1}{z^2}=\frac{1}{2}+\frac{z^2}{4!}+\cdots,$$

so taking the limit as $z\to 0$, we find that

$$\lim_{z\to 0}\frac{\cosh z-1}{z^2}=\frac{1}{2}.$$

If we define

$$f(z)=\begin{cases}\dfrac{\cosh z-1}{z^2}, & z\neq 0\\[2mm] \dfrac{1}{2}, & z=0,\end{cases}$$

the singularity at $z=0$ has been removed, and the resulting function is analytic for all z, so this function has a *removable singularity* at $z=0$.

(b) A partial fraction decomposition of $f(z)$ gives

$$f(z)=\frac{2}{z-1}+\frac{5}{(z+2)^2},$$

from which it can be seen that $f(z)$ has a *simple pole* at $z=1$ and a *pole of order* 2 at $z=-2$.

(c) As

$$f(z)=\frac{\sinh z}{z^5}=\frac{z+\frac{z^3}{3!}+\frac{z^5}{5!}+\frac{z^7}{7!}+\cdots}{z^5}=\frac{1}{z^4}+\frac{1}{3!}\frac{1}{z^2}+\frac{1}{5!}+\frac{1}{7!}z^2+\cdots,$$

the function is seen to have a *pole of order* 4 at the origin and to be analytic for all $z\neq 0$.

(d) Expanding the function gives

$$f(z)=z\exp(1/z)=z\left(1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\cdots\right)=z+1+\frac{1}{2!z}+\frac{1}{3!z^2}+\cdots,$$

showing that this function has an *isolated essential singularity* at the origin. ■

The Extended Complex Plane: The Point at Infinity

Unlike real numbers, complex numbers have no natural *order property*, so the inequality symbols $<$ and $>$ have no meaning when applied to complex numbers z_1 and z_2. However, $|z_1|$ and $|z_2|$ are real numbers that can be ordered, so this property can be used to give meaning to the "number" $z = \infty$. This is accomplished by saying that the complex sequence $\{z_n\}$ *tends to infinity*, written

$$\lim_{n\to\infty} z_n = \infty,$$

the meaning of the point at infinity in the complex plane, and the Riemann sphere

if

$$\lim_{n\to\infty} |z_n| = \infty.$$

This definition coincides with the corresponding one for real numbers, because the last result means that for any positive number L there is a positive integer N such that $|z_n| > L$ for all $n > N$. Thus, **the point at infinity** in the complex plane is taken to be the set of all points z such that $|z|$ lies outside the circle $|z| = L$ for any positive L. Accordingly, the set of all points outside a circle of arbitrarily large radius L centered on the origin is said to be a **neighborhood of infinity**. The complex plane, to which has been added the point at infinity is called the **extended complex plane**, and it is useful when performing various limiting operations.

A geometrical interpretation of $z = \infty$ that provides a justification for using the expression "point" at infinity can be obtained by making a stereographic projection of the extended complex plane onto a sphere. The concept, called the **Riemann sphere**, is illustrated in Fig. 15.5, which represents a sphere resting on the extended complex plane with its center above the origin. The point S of the sphere at the origin is called its *south pole* and the point N on its surface vertically above the origin is called its *north pole*.

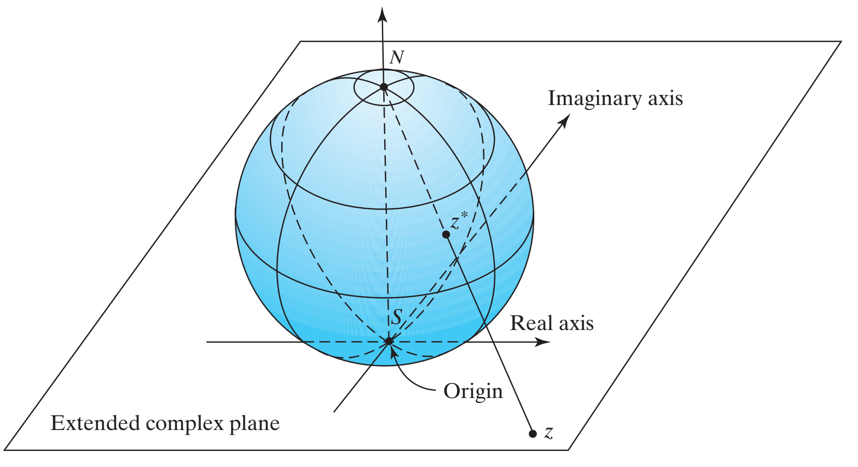


FIGURE 15.5 The Riemann sphere.

A point z on the extended complex plane is brought into correspondence with a point z^* on the sphere by taking z^* to be the point of intersection with the sphere of a straight line drawn from N to the point z. Each finite z corresponds to a unique point on the sphere, while all points in a neighborhood of $z = \infty$, which is outside a circle of arbitrarily large radius drawn in the extended complex plane with the origin as its center, correspond to an arbitrarily small neighborhood of N. Thus, the point N corresponds to the point at infinity in the extended complex plane. It is easy to see that circles in the extended complex plane with their center at the origin map to circles on the sphere (lines of latitude) while radial lines through S in the extended complex plane map to great circles (meridians) on the sphere (lines of longitude).

As already remarked, to study the behavior of a function $f(z)$ in a neighborhood of $z = \infty$, the substitution $z = 1/u$ is made, leading to an expression $F(u) = f(1/u)$. The behavior of $f(z)$ in a neighborhood of $z = \infty$ is then determined by the behavior of $F(u)$ in a neighborhood of $u = 0$.

Thus, if we consider the extended complex plane, the Laurent series for $f(z) = \frac{1}{z(1-z)}$ in a neighborhood of $z = \infty$ is obtained by setting $z = 1/u$, taking u to be arbitrarily small, and then, after expanding the result, writing $u = 1/z$. This leads to the result

$$F(u) = f(1/u) = \frac{1}{\frac{1}{u}\left(1 - \frac{1}{u}\right)} = \frac{u^2}{u-1}$$

$$= -u^2(1-u)^{-1} = -u^2(1 + u + u^2 + \cdots) = -\sum_{n=2}^{\infty} u^n,$$

and after substituting $u = 1/z$ this becomes the required Laurent series expansion in a neighborhood of $z = \infty$,

$$f(z) = -\sum_{n=2}^{\infty} \frac{1}{z^n} \quad \text{for } |z| > 1.$$

The same form of argument makes it possible to determine if a function $f(z)$ has a singularity at infinity and to classify such singularities. If we set $z = 1/u$ as before to obtain $F(u) = f(1/u)$, the singularity of $f(z)$ at $z = \infty$ is defined to be the same as that of $F(u)$ at $u = 0$.

For example, the function

$$f(z) = z^5 - \frac{1}{z^3}.$$

has a pole of order 3 at the origin in the ordinary complex plane, so to study its behavior at $z = \infty$ in the extended complex plane we set $z = 1/u$ when

$$F(u) = \frac{1}{u^5} - u^3,$$

showing that $F(u)$ has a pole of order 5 at $z = \infty$. Similarly, the function $f(z) = e^z$ is regular at the origin, that is, it has no singularity at the origin, but as $F(u) = f(1/u) = e^{1/u}$ we see that $f(z) = e^z$ has an essential singularity at $z = \infty$.

Summary

The Laurent series expansion of a function $f(z)$ about a singularity was defined, and it was shown that instead of using the formal definition to arrive at the expansion, it is often simpler to use a simple algebraic argument. Poles and singularities of functions were defined, and the meaning of the point at infinity in the complex plane was explained.

EXERCISES 15.3

In Exercises 1 through 12 find the Laurent series of $f(z)$ expanded about the given point and determine its annulus of convergence.

1. $f(z) = \dfrac{1}{z-2}$ expanded about $z = 0$.

2. $f(z) = \dfrac{1}{(z-a)^2}$ $(a \neq 0)$ expanded about $z = 0$.

3. $f(z) = \dfrac{1}{(z-a)(z-b)}$ $(0 < |a| < |b|)$, with $|z| < |a|$ expanded about $z = 0$.

4. $f(z) = \dfrac{1}{(z-a)(z-b)}$ $(0 < |a| < |b|)$, with $0 < |z-a| < |b-a|$ expanded about $z = a$.

5. $f(z) = \dfrac{1}{(z-a)(z-b)}$ $(0 < |a| < |b|)$, in the annulus $|a| < |z| < |b|$ when expanded about $z = 0$.

6. $f(z) = \dfrac{z^2 - 2z + 5}{(z-2)(z^2+1)}$ expanded about $z = 2$.

7. $f(z) = \exp\left(\dfrac{1}{1-z}\right)$ expanded (a) about $z = 1$, and (b) about $z = 0$ for $|z| > 1$.

8. $f(z) = \dfrac{1}{z(1-z)}$ expanded (a) about $z = 0$ and (b) about $z = 1$.

9. $f(z) = \sin\left(\dfrac{z}{1-z}\right)$ expanded about $z = 1$.

10. $f(z) = \dfrac{1}{(z-2)(z-3)}$ expanded about $z = 0$ for $|z| < 2$ and for $2 < |z| < 3$.

11. $f(z) = \dfrac{1}{(1-z)(z+2)}$ expanded about $z = 0$.

12. $f(z) = \dfrac{1}{z^2 + a^2}$ expanded about (a) $z = ia$ and (b) $z = 0$ for $|z| > |a|$.

In Exercises 13 through 16 find the first four terms of the Laurent series expansion of $f(z)$ about the given point.

13. $f(z) = \sinh\left(1 + \dfrac{1}{z}\right)$ expanded about $z = 0$.

14. $f(z) = \cosh\left(2 + \dfrac{1}{z}\right)$ expanded about $z = 0$.

15. $f(z) = \dfrac{\sin z \sin(z/3)}{z^3}$ expanded about $z = 0$.

16. $f(z) = \dfrac{\sin z \sinh(z/4)}{z^4}$ expanded about $z = 0$.

In Exercises 17 through 28 classify the nature of any singularities that occur in the finite complex plane.

17. $f(z) = \dfrac{1}{4z - z^3}$.

18. $f(z) = \dfrac{z}{1 + z^4}$.

19. $f(z) = \exp(-1/z^2)$.

20. $f(z) = \dfrac{1+z}{z(z^2+4)^2}$.

21. $f(z) = \dfrac{\sin z}{\sinh z}$.

22. $f(z) = \dfrac{1+z^2}{\cosh z}$.

23. $f(z) = \exp\left(\dfrac{z}{1-z}\right)$.

24. $f(z) = \cot(1/z)$.

25. $f(z) = \tan^2 z$.

26. $f(z) = \dfrac{\cos z}{z^2}$.

27. $f(z) = \dfrac{\cos 2z - 1}{\sin^2 z}$.

28. $f(z) = \dfrac{z^3 - 8z - 3}{z - 3}$.

Further Results

29. The integral for a_n in Theorem 15.13 defines the coefficients of the Laurent series for a function $f(z)$ expanded about the point z_0 that is convergent in the annulus $R_1 < |z - z_0| < R_2$. Use this integral to derive the **Cauchy inequalities for the coefficients of a Laurent series**

$$|a_n| \leq \frac{M}{R^n}, \quad \text{for } n = 0, \pm 1, \pm 2, \ldots,$$

where M is the greatest value of $|f(z)|$ on a circle $|z - z_0| = R$, with $R_1 < R < R_2$.

30. Use the result of Exercise 29 to show that if a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function such that when $|z| > R_1$ (the inner radius of the annulus of convergence in Theorem 15.13), and for a given nonnegative integer N,

$$|f(z)| < M|z|^N,$$

then $f(z)$ must be a polynomial of degree no greater than N.

In Exercises 31 through 34 find the Laurent series expansion of $f(z)$ in a neighborhood of $z = \infty$.

31. $f(z) = \dfrac{1}{z+3}$.

32. $f(z) = \dfrac{1}{(z^2+1)^2}$.

33. $f(z) = \mathrm{Log}\left(\dfrac{z^2}{1+z^2}\right)$.

34. $f(z) = \dfrac{1}{(z-a)(z-b)}$ $\times (0 < |a| < |b|)$.

In Exercises 35 through 40 determine the nature of the singularity of $f(z)$ at $z = \infty$.

35. $f(z) = \dfrac{1}{z - z^3}$.

36. $f(z) = \dfrac{z^5}{(1+z)^2}$.

37. $f(z) = \dfrac{1}{\sin z}$.

38. $f(z) = \dfrac{\cos 3z}{z^2}$.

39. $f(z) = e^{2iz}$.

40. $f(z) = \tan z$.

15.4 Residues and the Residue Theorem

Let an analytic function $f(z)$ have an isolated singularity at z_0. Then its Laurent series expansion about the point z_0,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n = \cdots + \frac{a_{-3}}{(z-z_0)^3} + \frac{a_{-2}}{(z-z_0)^{-2}} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots, \tag{22}$$

the residue and its connection with the Laurent expansion

will converge in some punctured disc $0 < |z - z_0| < R$. The **residue** of $f(z)$ at $z = z_0$, written $\operatorname{Res}[f(z), z_0]$, or simply $\operatorname{Res}[z_0]$ when there is no ambiguity about the function involved, is defined as the number a_{-1}, so that

$$\operatorname{Res}[f(z), z_0] = a_{-1}. \tag{23}$$

Thus, the residue of $f(z)$ at z_0 is the coefficient of the term $1/(z - z_0)$ in the principal part of its Laurent series expansion about z_0.

EXAMPLE 15.22

Find the residue of $f(z) = 1/(z^2 + 1)^2$ at the point $z = i$.

Solution It was shown in Example 15.19 that the Laurent series of $f(z) = 1/(z^2 + 1)^2$ expanded about the point $z = i$ is

$$f(z) = -\frac{1}{4(z-i)^2} - \frac{i}{4(z-i)} + \sum_{n=0}^{\infty} \frac{(n+3)i^n(z-i)^n}{2^{n+4}} \quad \text{for } 0 < |z-i| < 2,$$

so the residue at $z = i$ is seen to be

$$\operatorname{Res}[i] = -\frac{i}{4}.$$

From now on our concern will be with residues of analytic functions $f(z)$ whose only isolated singularities are poles. Then, if z_0 is a pole of $f(z)$ of order N, its Laurent series expansion about the pole will be of the form

$$f(z) = \frac{a_{-1}}{z-z_0} + \frac{a_{-2}}{(z-z_0)^2} + \cdots + \frac{a_{-N}}{(z-z_0)^N} + \sum_{n=0}^{\infty} a_n(z-z_0)^n, \tag{24}$$

where $a_{-N} \neq 0$, though some or all of the remaining coefficients $a_{-1}, a_{-2}, \ldots, a_{1-N}$ may vanish.

Let $f(z)$ be analytic at z_0. Then z_0 is a **zero** of the function $f(z)$ if $f(z_0) = 0$. In some neighborhood of z_0 the function will have a Taylor series expansion of the form

$$f(z) = \sum_{n=1}^{\infty} a_n(z-z_0)^n, \tag{25}$$

where to satisfy the condition $f(z_0) = 0$ we have set the coefficient $a_0 = 0$. The zero z_0 is called a **simple zero** of $f(z)$ if $a_1 \neq 0$, and a **zero of order** $\boldsymbol{N}$ if the first nonvanishing coefficient in (25) is a_N. If the zero is of order N we can write

a zero of order *n* and testing for a pole of order *n*

$$f(z) = (z-z_0)^N g(z), \tag{26}$$

where $g(z_0) \neq 0$ and $g(z)$ is analytic in a neighborhood of z_0, from which it follows that if $f(z)$ has a zero of order N at z_0, then $1/f(z)$ will have a pole of order N at z_0.

Inspection of (24) provides the following simple test for a pole of order N.

Test for a pole of order N

If $f(z)$ is analytic in the punctured disc $0 < |z - z_0| < R$, then a necessary and sufficient condition for it to have a pole of order N at z_0 is that

$$\lim_{z \to z_0} (z - z_0)^N f(z) = C, \qquad \text{where } C \neq 0.$$

In most cases, when z_0 is a pole of $f(z)$, it is simpler to determine the residue at z_0 by one of the formulas we will now derive than to develop the Laurent series expansion of $f(z)$ about z_0 and then to identify the residue with the coefficient a_{-1}.

The simplest case occurs when a function $f(z)$ of the form

$$f(z) = \frac{g(z)}{h(z)}$$

has a simple pole at z_0, and $g(z)$ and $h(z)$ are analytic functions in a neighborhood of z_0. Suppose first that $h(z)$ contains a factor $(z - z_0)$, and so can be written $h(z) = (z - z_0)F(z)$, where $F(z_0) \neq 0$. Then

$$f(z) = \frac{1}{(z - z_0)} \frac{g(z)}{F(z)},$$

but $H(z) = g(z)/F(z)$ is analytic at z_0 and so can be expanded in a Taylor series about z_0 of the form

$$H(z) = H(z_0) + (z - z_0)H'(z_0) + \frac{1}{2!}(z - z_0)^2 H''(z_0) + \cdots.$$

Using this result in the expression for $f(z)$ and writing $H(z_0) = g(z_0)/F(z_0)$ gives

$$f(z) = \frac{1}{z - z_0} \frac{g(z_0)}{F(z_0)} + H'(z_0) + \frac{1}{2!}(z - z_0)H''(z_0) + \cdots.$$

This shows that $\text{Res}[f(z), z_0]$, the coefficient of $1/(z - z_0)$ in the Laurent series expansion of $f(z)$ about z_0, is given by

$$\text{Res}[f(z), z_0] = \frac{g(z_0)}{F(z_0)}. \tag{27}$$

Now suppose that $f(z)$ is of the form

$$f(z) = \frac{g(z)}{h(z)}$$

and has a simple pole at z_0, but that $h(z)$ does *not* contain a factor $(z - z_0)$. Then, as $f(z)$ will have a Laurent series expansion about z_0 of the form

$$f(z) = \frac{g(z)}{h(z)} = \frac{\text{Res}[f(z), z_0]}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

we see that

$$\text{Res}[f(z), z_0] = \lim_{z \to z_0} \left\{ \frac{(z - z_0)g(z)}{h(z)} \right\}.$$

Using the fact that $h(z_0) = 0$ allows this to be written

$$\mathrm{Res}[f(z), z_0] = \lim_{z\to z_0}\left\{g(z)\Big/\left(\frac{h(z)-h(z_0)}{z-z_0}\right)\right\},$$

but $h(z)$ is analytic and

$$h'(z_0) = \lim_{z\to z_0}\left(\frac{h(z)-h(z_0)}{z-z_0}\right),$$

so

$$\mathrm{Res}[f(z), z_0] = \frac{g(z_0)}{h'(z_0)}. \tag{28}$$

Finally we consider the case where $f(z)$ has a pole of order N at z_0, and so has the Laurent series expansion about z_0 given by (24). Multiplying (24) by $(z-z_0)^N$ gives

$$(z-z_0)^N f(z) = a_{-1}(z-z_0)^{N-1} + a_{-2}(z-z_0)^{N-2} + \cdots + a_{-N} + \sum_{n=0}^{\infty} a_n(z-z_0)^{N+n},$$

and after differentiating this with respect to z we find that

$$\frac{d}{dz}[(z-z_0)^N f(z)] = (N-1)a_{-1}(z-z_0)^{N-2} + (N-2)a_{-2}(z-z_0)^{N-3} + \cdots + a_{1-N} + \sum_{n=0}^{\infty}(N+n)a_n(z-z_0)^{N+n-1}.$$

Taking the limit of this result as $z \to z_0$ reduces it to

$$\lim_{z\to z_0}\left\{\frac{d}{dz}[(z-z_0)^N f(z)]\right\} = a_{1-N}.$$

A repetition of this process yields the formula

$$\lim_{z\to z_0}\left\{\frac{d^2}{dz^2}[(z-z_0)^N f(z)]\right\} = a_{2-N},$$

so as $\mathrm{Res}[f(z), z_0] = a_{-1}$, after $N-1$ differentiations this same form of argument brings us to the final result

$$\mathrm{Res}[f(z), z_0] = \frac{1}{(N-1)!}\lim_{z\to z_0}\left\{\frac{d^{N-1}}{dz^{N-1}}[(z-z_0)^N f(z)]\right\}. \tag{29}$$

formulas for finding the residue at a simple pole and at a pole of order *n*

Taken together, results (27) to (29) have established the following formulas for the calculation of residues.

Formulas for the residue at a pole of a function of the form $f(z) = g(z)/h(z)$

1. **(i)** Let a function $f(z)$ that is analytic in a punctured disc $0 < |z-z_0| < R$ have a simple pole at z_0. Then if $f(z) = g(z)/h(z)$, and $h(z)$ contains a factor $(z-z_0)$ and so can be written $h(z) = (z-z_0)F(z)$ where $F(z_0) \neq 0$, the residue of $f(z)$ at z_0 is given by the formula

$$\boxed{\mathrm{Res}[f(z), z_0] = \frac{g(z_0)}{F(z_0)},} \tag{30}$$

(ii) and if $h(z_0) = 0$, but $h(z)$ does not necessarily contain a factor $(z - z_0)$, the residue of $f(z)$ at z_0 is given by the formula

$$\text{Res}[f(z), z_0] = \frac{g(z_0)}{h'(z_0)}. \tag{31}$$

2. Finally, if $f(z)$ has a pole of order N at z_0, the residue of $f(z)$ at z_0 is given by the formula

$$\text{Res}[f(z), z_0] = \frac{1}{(N-1)!}\lim_{z\to z_0}\left\{\frac{d^{N-1}}{dz^{N-1}}[(z-z_0)^N f(z)]\right\}. \tag{32}$$

EXAMPLE 15.23

Find the residues at the poles of the functions
(a) $f(z) = \dfrac{z^2+2z+3}{z-i}$, (b) $f(z) = \dfrac{1}{(z^2+1)^2}$, (c) $f(z) = \dfrac{1}{z\sin z}$,
(d) $f(z) = \text{sech}\, z$.

Solution

(a) $f(z)$ has a simple pole at $z = i$, with $g(z) = z^2 + 2z + 3$ and $h(z) = z - i$, so as the denominator contains the factor $(z - i)$, making use of (30) gives

$$\text{Res}[i] = \left[(z-i)\frac{(z^2+2z+3)}{(z-i)}\right]_{z=i} = [z^2+2z+3]_{z=i} = 2(1+i).$$

(b) The function has poles of order 2 at $z = \pm i$, so from (32) with $N = 2$ we see that

$$\text{Res}[i] = \frac{1}{1!}\lim_{z\to i}\left\{\frac{d}{dz}\left[(z-i)^2\frac{1}{(z^2+1)^2}\right]\right\} = -\frac{i}{4},$$

and similarly

$$\text{Res}[-i] = \frac{1}{1!}\lim_{z\to -i}\left\{\frac{d}{dz}\left[(z+i)^2\frac{1}{(z^2+1)^2}\right]\right\} = \frac{i}{4}.$$

This simple calculation for the determination of Res$[i]$ should be compared with the extensive calculations needed to arrive at the full Laurent series for $f(z)$ expanded about the point $z = i$ in Example 15.22, where the coefficient of the term $1/(z - i)$ was, of course, equal to $-i/4$.

(c) The function has poles at the zeros of the denominator $z\sin z$. For small z

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right),$$

so near the origin

$$f(z) = \frac{1}{z^2}\cdot\frac{1}{\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)},$$

showing that $f(z)$ has a pole of order 2 at the origin. Elsewhere, $z \neq 0$ and the factor $\sin z$ has zeros at $\pm n\pi$ for $n = 0, 1, 2, \ldots,$ corresponding to simple poles of $f(z)$.

The residue at the origin, obtained from (32) with $N = 2$ and $z_0 = 0$, is

$$\mathrm{Res}[0] = \frac{1}{1!}\lim_{z\to 0}\left\{\frac{d}{dz}\left[z^2\frac{1}{z\sin z}\right]\right\} = \lim_{z\to 0}\left\{\frac{\sin z - z\cos z}{\sin^2 z}\right\}.$$

This is an indeterminate form, so applying l'Hôpital's rule we find that

$$\mathrm{Res}[0] = \lim_{z\to 0}\left\{\frac{z\sin z}{2\sin z\cos z}\right\} = 0.$$

The residues at the simple zeros $\pm n\pi$, for $n = 1, 2, \ldots$, follow by setting $g(z) = 1$ and $h(z) = z\sin z$ in (31) to obtain

$$\mathrm{Res}[\pm n\pi] = [1/(\sin z + z\cos z)]_{z=\pm n\pi} = \frac{\pm(-1)^n}{n\pi}, \quad \text{for } n = 1, 2, \ldots.$$

(d) Writing $f(z) = 1/\cosh z$ shows that poles of $f(z)$ are located at the zeros $(2n+1)\pi i/2$ of $\cosh z$ for $n = 0, \pm 1, \pm 2, \ldots$. So $f(z)$ has simple poles at $z = (2n+1)\pi i/2$ for $n = 0, \pm 1, \pm 2, \ldots$. From (31) using $g(z) = 1$ and $h(z) = \cosh z$ we have

$$\mathrm{Res}[(2n+1)\pi i] = \frac{1}{\sinh\{(2n+1)\pi i/2\}} = \frac{1}{i\sin\{(2n+1)\pi/2\}} = (-1)^{n+1} i,$$

for $n = 0, \pm 1, \pm 2, \ldots$. ■

When the limit in (32) is difficult to evaluate, it is necessary to determine the residue by developing the Laurent series expansion to the point where the coefficient of the term $1/(z - z_0)$ can be identified. This situation is illustrated in the next example.

EXAMPLE 15.24 Find the residue of

$$f(z) = \sin\left(\frac{z}{z+1}\right).$$

Solution Inspection of the argument of the sine function shows that its only singularity occurs at $z = -1$, but the function is sufficiently complicated that result (32) is not useful. Accordingly, to find the coefficient of the term $1/(z+1)$ in the Laurent series expansion about $z = -1$, we rewrite $f(z)$ as

$$f(z) = \sin\left(1 - \frac{1}{z+1}\right),$$

and then use the familiar trigonometric identity $\sin(A - B) = \sin A\cos B - \cos A\sin B$ to expand this as

$$f(z) = \sin(1)\cos\left(\frac{1}{z+1}\right) - \cos(1)\sin\left(\frac{1}{z+1}\right).$$

Replacing the cosine and sine function involving z with the first few terms of their Maclaurin series gives

$$f(z) = \sin(1)\left(1 - \frac{1}{2!(z+1)^2} + \frac{1}{4!(z+1)^4} - \cdots\right)$$
$$-\cos(1)\left(\frac{1}{z+1} - \frac{1}{3!(z+1)^3} + \frac{1}{5!(z+1)^5} - \cdots\right).$$

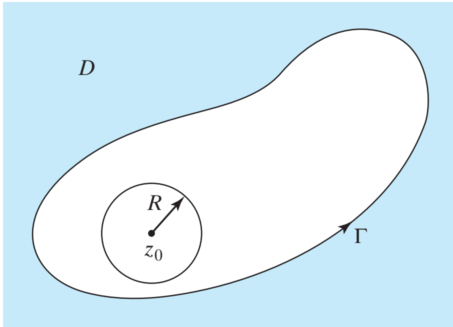


FIGURE 15.6 A contour Γ containing a point z_0 at which $f(z)$ has a pole.

Inspection then shows that the coefficient of the term $1/(z+1)$ is $-\cos(1)$, so

$$\text{Res}[f(z), -1] = -\cos(1).$$

why residues are important

The crucial importance of residues in the theory of complex integration follows from the fact that when a function $f(z)$ has a pole of any order at a point z_0 in a domain D, but is analytic elsewhere D, then the integral around any contour in D that contains the pole at z_0 depends *only* on the value of the residue at z_0. To prove this assertion, and to find the value of the integral, we consider the case in which $f(z)$ has a pole of order N at a point z_0 in a domain D but is analytic elsewhere in D.

We take a positively oriented contour Γ in D as shown in Fig. 15.6, represent $f(z)$ by its Laurent series (24) expanded about z_0, and integrate the result around Γ. As a result we have

$$\int_\Gamma f(z)dz = \int_\Gamma \left(\frac{a_{-1}}{z-z_0} + \frac{a_{-2}}{(z-z_0)^2} + \cdots + \frac{a_{-N}}{(z-z_0)^N}\right)dz + \sum_{n=0}^{\infty} a_n \int_\Gamma (z-z_0)^n dz, \tag{33}$$

where term-by-term integration of the infinite series at the right is allowed by virtue of Theorem 15.12.

It was shown in Example 14.4 that

$$\int_{|z-z_0|=R} (z-z_0)^n dz = 0 \quad \text{for } n = -2, -3, \ldots, \quad \text{and} \quad n = 0, 1, 2, \ldots,$$

where the circle $|z - z_0| = R$ lies within D. The deformation of contour theorem asserts that these results are true for any contour Γ in D that contains z_0, as a result of which (33) reduces to

$$\int_\Gamma f(z)dz = \int_\Gamma \frac{a_{-1}}{z-z_0}dz,$$

and so to the equivalent result

$$\int_\Gamma f(z)dz = \text{Res}[z_0] \int_\Gamma \frac{dz}{z-z_0}.$$

In Example 14.5 it was shown that

$$\int_{|z-z_0|=R} \frac{dz}{z-z_0} = 2\pi i,$$

when the circle $|z-z_0| = R$ lies within D. The deformation of contour theorem allows this result to remain true when the circle $|z-z_0| = R$ is replaced by the contour Γ containing z_0, so we have proved the extremely important result that

$$\int_\Gamma f(z)dz = 2\pi i \operatorname{Res}[f(z), z_0]. \tag{34}$$

This result is easily extended to the case of a function $f(z)$ with m poles in D located at the points $z_1, z_2, \ldots, z_m$. To see this, let the poles in D lie inside a simple positively oriented closed contour Γ contained in D, and enclose the pole at z_r in a small positively oriented circle Γ_r lying inside D, with $r = 1, 2, \ldots, m$. Integrating around Γ and using the extended Cauchy–Goursat theorem, we obtain

$$\int_\Gamma f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz + \cdots + \int_{\Gamma_m} f(z)dz,$$

but from (34),

$$\int_{\Gamma_r} f(z)dz = 2\pi i \operatorname{Res}[f(z), z_r], \quad \text{for } r = 1, 2, \ldots, m,$$

so

$$\int_\Gamma f(z)dz = 2\pi i[\operatorname{Res}[f(z), z_1] + \operatorname{Res}[f(z), z_2] + \cdots + \operatorname{Res}[f(z), z_m]]. \tag{35}$$

This result contains the Cauchy–Goursat theorem as a special case, because if the contour Γ in D contains no poles of $f(z)$, the function has no residues inside Γ and so

$$\int_\Gamma f(z)dz = 0.$$

The fundamental result contained in (35) forms our next theorem.

THEOREM 15.14

contour integrals and the residue theorem

The residue theorem Let $f(z)$ have poles at $z_1, z_2, \ldots, z_m$ in a domain D and be analytic elsewhere in D. Then if Γ is any simple positively oriented contour in D containing the points $z_1, z_2, \ldots, z_m$,

$$\int_\Gamma f(z)dz = 2\pi i \sum_{r=1}^{m} \operatorname{Res}[f(z), z_r].$$

■

evaluating a contour integral using residues

Expressed in words, this theorem says that the integral of $f(z)$ around Γ is $2\pi i$ times the sum of the residues enclosed in Γ. The next example illustrates the application of Theorem 15.14 to a function with three poles.

EXAMPLE 15.25

Find all the residues of the function $f(z) = \frac{e^z}{(z+2i)^3(z^2-4)}$, and use them to determine $\int_\Gamma f(z)dz$ around the following positively oriented contours in which Γ is (a) the circle Γ_1 given by $|z+3i| = 2$, (b) the circle Γ_2 given by $|z-2| = 1$, and (c) the circle Γ_3 given by $|z| = 4$.

Solution Inspection of $f(z)$ shows it has a pole of order 3 at $z=-2i$, and simple poles at $z=\pm 2$. Applying (32) to find the residue at $z=-2i$ gives

$$\operatorname{Res}[f(z),-2i]=\frac{1}{2!}\left\{\frac{d^2}{dz^2}\left[(z+2i)^3\frac{e^z}{(z+2i)^3(z^2-4)}\right]\right\}_{z=-2i}$$
$$=\frac{1}{2}\left\{\frac{d^2}{dz^2}\left[\frac{e^z}{z^2-4}\right]\right\}_{z=-2i}=\frac{e^{-2i}}{16}\left(i-\frac{3}{4}\right),$$

and as the poles at $z=\pm 2$ are only simple poles, it follows from (30) that

$$\operatorname{Res}[f(z),-2]=\left[\frac{e^z}{(z+2i)^3(z-2)}\right]_{z=-2}=\frac{e^{-2}}{128}(i-1),$$

and

$$\operatorname{Res}[f(z),2]=\left[\frac{e^z}{(z+2i)^3(z+2)}\right]_{z=2}=-\frac{e^2}{128}(1+i).$$

The three contours Γ_1, Γ_2, and Γ_3 and the location of the poles of $f(z)$ are shown in Fig. 15.7. Only the pole of order 3 at $z=-2i$ lies inside contour Γ_1, and only the simple pole at $z=2$ lies inside contour Γ_2, though all three poles lie inside contour Γ_3.

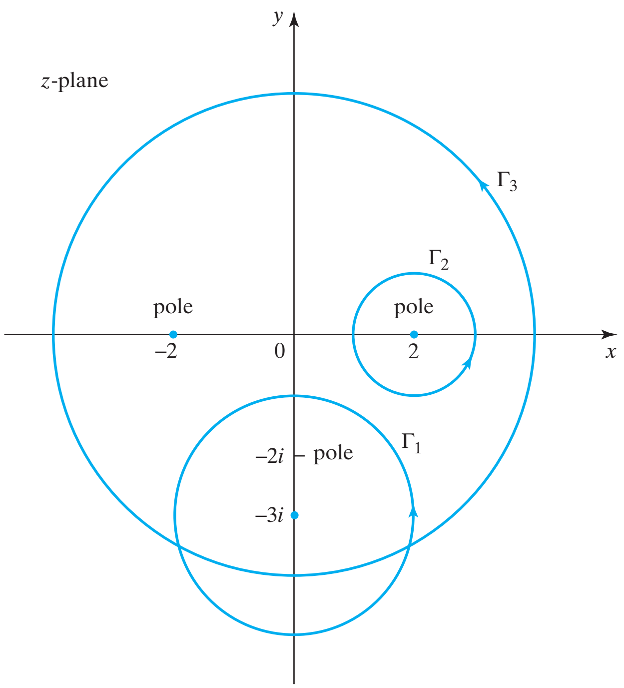


FIGURE 15.7 The contours Γ_1, Γ_2, and Γ_3 and the location of the poles of $f(z)$.

Applying Theorem 15.14 we have

$$\int_{\Gamma_1} f(z)dz = 2\pi i \left(\frac{e^{-2i}}{16}\left(i - \frac{3}{4}\right)\right) = -\frac{\pi e^{-2i}}{8}\left(1 + \frac{3i}{4}\right)$$

$$\int_{\Gamma_2} f(z)dz = 2\pi i \left(-\frac{e^{2}(1+i)}{128}\right) = \frac{\pi e^{2}(1-i)}{64}$$

and

$$\int_{\Gamma_3} f(z)dz = 2\pi i \left\{-\frac{e^{2}(1+i)}{128} + \frac{e^{-2}(i-1)}{128} + \frac{e^{-2i}}{16}\left(i - \frac{3}{4}\right)\right\}$$

$$= \frac{\pi}{64}(e^2 - e^{-2}) - \frac{\pi e^{-2i}}{8} - \frac{i\pi}{64}(6e^{-2i} + e^2 + e^{-2}). \qquad ■$$

EXAMPLE 15.26 Find

$$\int_{|z+1|=1} \sin\left(\frac{z}{z+1}\right) dz.$$

Solution We saw in Example 15.24 that the only singularity of the integrand is a simple pole at $z = -1$ with residue $-\cos(1)$. So as the circle $|z+1| = 1$ contains the pole, it follows immediately from Theorem 15.14 that

$$\int_{|z+1|=1} \sin\left(\frac{z}{z+1}\right) dz = 2\pi i\{-\cos(1)\} = -2\pi i \cos(1). \qquad ■$$

Summary

The Laurent series was used to introduce the idea of a residue, and formulas for finding the residue at a simple pole and at a pole of order *n* were derived. The relationship of residues to contour integrals was explained, and the fundamental residue theorem was proved.

EXERCISES 15.4

In Exercises 1 through 16 find the residues of the given functions at their poles in the finite complex plane.

1. $f(z) = \dfrac{z+3}{z^2-4}$.
2. $f(z) = \dfrac{z^2+1}{z^2(z+2)}$.
3. $f(z) = \dfrac{z^2+z-2}{z^2(z+1)}$.
4. $f(z) = \dfrac{z^2+1}{z(z+1)^3}$.
5. $f(z) = \dfrac{\sin z}{z^2(z-1)}$.
6. $f(z) = \dfrac{\cos z}{z^2-5z+6}$.
7. $f(z) = \dfrac{z^2+3}{\sin z}$.
8. $f(z) = \dfrac{\sin 3z}{(z-1)^4}$.
9. $f(z) = \tan z$.
10. $f(z) = \cot z$.
11. $f(z) = \dfrac{1}{e^z+1}$.
12. $f(z) = \dfrac{\sinh z}{\sin z}$.
13. $f(z) = \dfrac{\sin z}{\sinh z}$.
14. $f(z) = \dfrac{\pi}{z^2 \tan \pi z}$.
15. $f(z) = \cos\left(\dfrac{1}{z-2}\right)$.
16. $f(z) = z^3 \cos\left(\dfrac{1}{z-2}\right)$.

Evaluate the contour integrals in Exercises 17 through 28.

17. $\displaystyle\int_{|z|=1} \frac{\sin z}{z^4} dz$.
18. $\displaystyle\int_{|z|=2} \frac{\cos z}{z^2} dz$.
19. $\displaystyle\int_{|z|=1} \frac{z^2+1}{z(z-6)} dz$.
20. $\displaystyle\int_{|z-2|=1/2} \frac{z\,dz}{(z-1)(z-2)^2}$.
21. $\displaystyle\int_{|z-1|=1} \frac{dz}{z^4+1}$.
22. $\displaystyle\int_{|z|=2} \frac{dz}{(z-3)(z^5-1)}$.
23. $\displaystyle\int_{|z|=1} \frac{e^z}{z^2(z^2-9)} dz$.
24. $\displaystyle\int_{|z|=1/2} z^n e^{2/z} dz (n = 0, \pm 1, \pm 2, \ldots)$.

25. $\displaystyle\int_{|z-i|=1} \frac{1-e^{2iz}}{z^2+1} dz.$

26. $\displaystyle\int_{|z|=2} \frac{\cos z}{z^3} dz.$

27. $\displaystyle\int_{|z|=2} (2z-1)\cos\left(\frac{z}{z-1}\right) dz.$

28. $\displaystyle\int_{|z|=4} \frac{e^{1/(z-1)}}{z-2} dz.$

Integrals of the form

$$\int_0^{2\pi} \text{Rational}[\cos\theta, \sin\theta] d\theta,$$

where Rational$[\cos\theta, \sin\theta]$ is a rational function of $\cos\theta$ and $\sin\theta$ (a quotient of polynomials in $\cos\theta$ and $\sin\theta$), can be evaluated by making the substitutions

$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right), \quad \sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right), \quad \text{and} \quad d\theta = \frac{dz}{iz},$$

which all follow from De Moivre's theorem, and then integrating around the unit circle $|z| = 1$. Use this approach to evaluate the trigonometric integrals in Exercises 29 through 33.

29. $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+\cos\theta} \quad (a > 1).$

30. $\displaystyle\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} \quad (a > 1).$

31. $\displaystyle\int_0^{2\pi} \frac{d\theta}{3+\sin\theta}.$

32. $\displaystyle\int_0^{2\pi} \frac{d\theta}{3-2\sin\theta}.$

33. $\displaystyle\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$ $(0 < a < 1)$.

34. Prove that if $f(z) = g(z)/h(z)$ is the quotient of two functions where $g(z)$ is analytic at z_0 with $g(z_0) \neq 0$, and $h(z)$ has a zero of order 2 at z_0, then

$$\text{Res}[f(z), z_0] = \frac{6g'(z_0)h''(z_0) - 2g(z_0)h'''(z_0)}{3[h''(z_0)]^2}.$$

15.5 Evaluation of Real Integrals by Means of Residues

The previous section showed how the residues at the poles of an analytic function inside a simple closed contour determine the value of integral of the function around the contour. In the present section we show how by taking some part of the contour along the real axis it is possible to use the method of residues to evaluate improper real integrals of the form

$$\int_0^{\infty} f(x)dx \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)dx,$$

where $f(x)$ may become infinite at a finite number of points in the interval of integration.

(a) Convergence, Divergence, and Cauchy Principal Values of Integrals

The meaning of integration over a semi-infinite or an infinite interval obtained by complex analysis needs to be explained. It will be recalled from elementary calculus that when $f(x)$ remains finite over the interval of integration, the values of these improper integrals are defined as the limiting values

$$\int_0^{\infty} f(x)dx = \lim_{R\to\infty} \int_0^{R} f(x)dx \quad \text{and}$$
$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R_1\to\infty, R_2\to\infty} \int_{-R_1}^{R_2} f(x)dx, \tag{36}$$

where in the second integral R_1 and R_2 are allowed to tend to infinity *independently* of each other. If these limiting values are finite, the improper integrals are said to

converge to the values of their respective limits, and they are said to be *divergent* if the limits are undefined or are infinite.

If, in addition, $f(x)$ becomes infinite at a point x_0 inside the interval of integration, say in the first of these integrals, the value of the integral is to be interpreted as

$$\int_0^\infty f(x)dx = \lim_{\alpha\to 0}\int_0^{x_0-\alpha} f(x)dx + \lim_{\beta\to 0,\, R\to\infty}\int_{x_0+\beta}^{R} f(x)dx, \tag{37}$$

where $\alpha > 0$ and $\beta > 0$ are allowed to tend to zero *independently* of each other. If this limit is finite, the integral is said to *converge* to the value of the limit, and it is said to be *divergent* if the limit is undefined or infinite. A corresponding interpretation applies to integrals over the interval $(-\infty, \infty)$ when $f(x)$ is infinite at a point x_0 inside the interval of integration. If $f(x)$ is infinite at several points inside the interval of integration, the limiting operation shown in (37) is extended in an obvious manner.

Improper integrals such as (37) can occur that are *divergent* if α and β are allowed to tend to zero independently of each other, but are *convergent* if $\beta = \alpha$ as $\alpha \to 0$. In convergent integrals of this type the upper limit of integration in the first integral in (37) is $x_0 - \alpha$ and the lower limit in the second integral is $x_0 + \alpha$. Similarly, improper integrals over infinite intervals such as the second integral in (36) occur that are divergent when R_1 and R_2 are allowed to tend to infinity independently of each other, but are convergent if $R_1 = R_2$, as $R_1 \to +\infty$.

The value of an improper integral when the limits of integration on either side of an infinity of the integrand at x_0 are of the form $x_0 - \alpha$ and $x_0 + \alpha$ as $\alpha \to 0$, and when the integral is over the infinite interval $(-\infty, \infty)$ the upper and lower limits of integration are of the form $R_1 = R_2$, as $R_1 \to +\infty$, is called the **Cauchy principal value** of the integral. The Cauchy principal value of an integral is indicated by inserting the symbol P.V. in front of the integral sign. So, if in the integral of $f(x)$ over the interval $[0, \infty)$, the function $f(x)$ has an infinity at x_0, its Cauchy principal value is defined as

Cauchy principal value

$$\text{P.V.}\int_0^\infty f(x)dx = \lim_{\alpha\to 0}\int_0^{x_0-\alpha} f(x)dx + \lim_{\alpha\to 0,\, R\to\infty}\int_{x_0+\alpha}^{R} f(x)dx \quad (\alpha > 0). \tag{38}$$

In some improper integrals of the type shown in the second expression in (36), allowing R_1 and R_2 to approach infinity at different rates produces the same result as the Cauchy principal value, and when this occurs the symbol P.V. can be dropped. This happens, for example, with the integral

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \lim_{R_1\to\infty,\, R_2\to\infty}\int_{-R_1}^{R_2}\frac{dx}{1+x^2} = \lim_{R_1\to\infty,\, R_2\to\infty}\{\text{Arctan } R_2 - \text{Arctan}(-R_1)\}$$
$$= \left\{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right\} = \pi,$$

because it is also true that

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{1+x^2} = \lim_{R\to\infty}\{\text{Arctan } R - \text{Arctan}(-R)\} = \pi.$$

This is an integral for which

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi.$$

As the integrand is an even function of x, these results allow us to conclude that

$$\int_{0}^{\infty} \frac{dx}{1+x^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}.$$

The situation is quite different in the case of the integral

$$\int_{-\infty}^{\infty} \sin x dx,$$

because although $\sin x$ is continuous and bounded for all x the integral is divergent. This result follows from the fact that

$$\lim_{R_1\to\infty, R_2\to\infty} \int_{-R_1}^{R_2} \sin x dx = \lim_{R_1\to\infty, R_2\to\infty} \{\cos R_2 - \cos R_1\},$$

so the limit is not defined, though the Cauchy principal value of the integral is finite because

$$\text{P.V.}\int_{-\infty}^{\infty} \sin x dx = \lim_{R\to\infty} \int_{-R}^{R} \sin x dx = \lim_{R\to\infty} \{\cos R - \cos(-R)\} = 0.$$

Another example of a divergent integral for which the Cauchy principal value is finite is

$$\int_{-\infty}^{\infty} \frac{x}{1+x^2} dx.$$

The divergence of the integral follows from the fact that

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{x}{1+x^2} dx &= \lim_{R_1\to\infty, R_2\to\infty} \int_{-R_1}^{R_2} \frac{x}{1+x^2} dx \\ &= \lim_{R_1\to\infty, R_2\to\infty} \frac{1}{2}\left\{\ln\left(1+R_2^2\right) - \ln\left(1+R_1^2\right)\right\},\end{aligned}$$

because this limit is not defined if $R_1 \neq R_2$. When $R_1 = R_2$ the Cauchy principal value follows from the preceding result, from which it is seen to be zero, so we say

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{x}{1+x^2} dx = 0.$$

a comparison test for improper integrals

Tests exist that enable the convergence or divergence of various types of improper integral to be established without the need for direct integration, and these are necessary because in most cases it is either difficult or impossible to evaluate the integral analytically. The simplest of these tests, called a **comparison test**, establishes the convergence (or divergence) of an improper integral by comparing its integrand with the integrand of an improper integral whose convergence or divergence properties are known. Thus, if, for example, the improper integral $\int_{-\infty}^{\infty} g(x)dx$ is known to be convergent, and $f(x)$ is such that $0 \le f(x) \le g(x)$, then the improper integral $\int_{-\infty}^{\infty} f(x)dx$ is convergent. This follows because then the integral $\int_{-\infty}^{\infty} f(x)dx$ is

bounded by

$$0 \le \int_{-\infty}^{\infty} f(x)dx \le \int_{-\infty}^{\infty} g(x)dx.$$

If, however, the improper integral $\int_{-\infty}^{\infty} g(x)dx$ is known to be divergent and $0 \le g(x) \le f(x)$, then

$$0 \le \int_{-\infty}^{\infty} g(x)dx \le \int_{-\infty}^{\infty} f(x)dx,$$

showing that the integral $\int_{-\infty}^{\infty} f(x)dx$ is divergent. Different forms of comparison tests exist, and corresponding tests apply to improper integrals over the interval $[0, \infty)$.

The concept of the Cauchy principal value of an integral is important when evaluating real improper integrals by means of contour integration, and especially when the integrand has an infinity at one or more points inside the interval of integration. This is because the method of evaluating such integrals gives rise automatically to the Cauchy principal value of the integral. Whether a real improper integral determined by contour integration also exists in the sense of (36) or (37), thereby allowing the symbol P.V. to be dropped from in front of the integral, must be determined separately.

(b) Improper Integrals of Rational Functions without Poles on the Real Axis

integrals of rational functions without poles on the real axis

As improper real integrals only involve integration along the real axis, in order to evaluate them by contour integration a suitable simple closed contour Γ must be introduced that includes as part of the contour the piece of the real axis that is involved. An essential feature of an analytic function $f(z)$ that is to be integrated must be that it, or its real or imaginary part, reduces to the required real improper integral on the real axis. In addition to this, in general, on the segment of the contour Γ that does not include the real axis, the modulus of $f(z)$ must tend to zero sufficiently rapidly as $|z| \to \infty$ that the integral around that segment vanishes.

When the entire real axis is involved, the contour Γ is usually taken to be the contour formed by the segment of the real axis from $-R$ to R, and the semicircle Γ_R with the equation $|z| = R$ in the upper half of the complex plane, with the sense of integration taken in the counterclockwise sense around Γ, as shown in Fig. 15.8a.

If we consider functions $f(z)$ that have no poles on the real axis, an improper integral of $f(z)$ over the interval $(-\infty, \infty)$ is evaluated by first taking R sufficiently large that all the poles of $f(z)$ in the upper-half of the complex plane lie inside Γ, applying the residue theorem to $\int_\Gamma f(z)dz$, and then proceeding to the limit at $R \to \infty$. It is this choice of contour that introduces the Cauchy principal value of improper integrals taken over an infinite interval.

indenting a contour

Later we will consider the situation in which a simple pole of $f(z)$ occurs on the real axis at x_0, when we will see it is necessary to exclude it from the contour of Γ by **indenting** the contour at x_0 by the addition of a small semicircle of radius

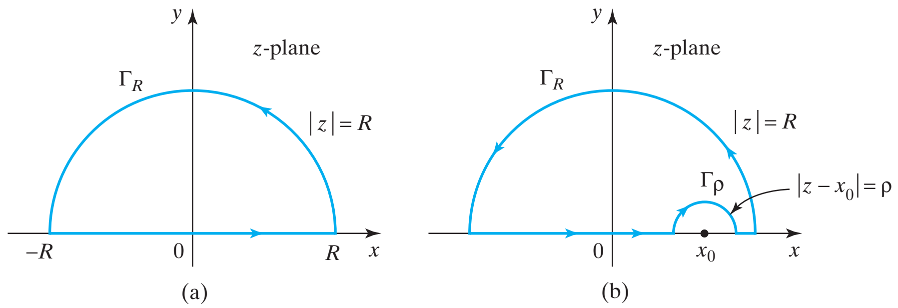


FIGURE 15.8 (a) The contour Γ in the upper half of the complex plane. (b) An indented contour Γ in the upper half of the complex plane.

ρ extending into the upper half of the complex plane, as shown in Fig. 15.8b. Then, after applying the residue theorem and giving due consideration to the effect of integration around the indentation, R is allowed to tend to infinity and ρ to tend to zero. In such a case the Cauchy principal value of the integral is due to reducing the indentation at x_0 to one of vanishingly small radius, and also to taking the limit symmetrically with respect to the origin as R tends to infinity.

This general approach to the evaluation of real integrals will be seen to work for functions $f(z)$ with the property that the integral around the semicircular part of the contour Γ_R vanishes in the limit as the radius of the semicircle $R \to \infty$. This means that for the method to succeed we must impose the condition

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z)dz = 0. \tag{39}$$

Later we will find conditions to be satisfied by the most frequently occurring types of integrand for which this result is always true. First, however, to illustrate the general approach, we begin by assuming condition (39) and applying the method to a typical example.

EXAMPLE 15.27 Evaluate the integral

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{1+x^4},$$

and show that the P.V. symbol can be omitted from the result.

Solution The function $f(z) = 1/(1+z^4)$ reduces to $f(x) = 1/(1+x^4)$ on the real axis, and the integrand has simple poles at the four zeros of $1+z^4$ given by $z_k = e^{i\pi(1+2k)/4}$ with $k = 0, 1, 2, 3$, but only the two zeros at

$$z_0 = e^{\pi i/4} \quad \text{and} \quad z_1 = e^{3\pi i/4}$$

lie in the upper half of the complex plane. So, as the interval of integration extends over the entire real axis, we will consider the integral of $f(z)$ around the contour of Fig. 15.8a.

A simple calculation shows that

$$\operatorname{Res}[f(z), z_0] = -\frac{1}{4}e^{i\pi/4} \quad \text{and} \quad \operatorname{Res}[f(z), z_1] = \frac{1}{4}e^{-i\pi/4},$$

so when the radius R of the semicircle Γ_R in Fig. 15.8a is large enough for the poles at z_0 and z_1 to lie inside Γ, an application of the residue theorem gives

$$\int_\Gamma \frac{dz}{1+z^4} = \int_{-R}^{R} \frac{dx}{1+x^4} + \int_{\Gamma_R} \frac{dz}{1+z^4} = \{\operatorname{Res}[f(z), z_0] + \operatorname{Res}[f(z), z_1]\}.$$

Letting $R \to \infty$, assuming that $\lim_{R\to\infty} \int_{\Gamma_R} \frac{dz}{1+z^4} = 0$, and substituting the values of the residues reduce this to

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \pi \left(\frac{e^{i\pi/4} - e^{-i\pi/4}}{2i} \right) = \pi \sin \frac{\pi}{4} = \frac{\pi}{\sqrt{2}}.$$

The symbol P.V. can only be omitted if the Cauchy principal value and the value of the improper integral are equal. This result will be true if we can show that the improper integral converges, because the Cauchy principal value is obtained as one of the possible ways in which the limits in (36) may be taken, so that then the two integrals must be equal.

We use a comparison argument to justify the removal of the P.V. symbol. The integrand $1/(1+x^4) \leq 1/(1+x^2)$ for all x, so the convergence of the integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ that has been established proves the convergence of $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$, and so justifies writing

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}.$$

The integrand is finite, continuous, and symmetric about the origin, so we may conclude that

$$\int_0^{\infty} \frac{dx}{1+x^4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}.$$ ■

The theorem we now prove provides conditions that ensure the validity of the limit in (39) when the modulus of the integrand $f(z)$ decreases sufficiently rapidly as $|z|$ becomes large. The theorem is particularly useful when $f(z)$ is a quotient of two polynomials in z, that is to say when $f(z)$ is a **rational function**, which we choose to write as

$$f(z) = \frac{a_0 + a_1 z + \cdots + a_m z^m}{b_0 + b_1 z + \cdots + b_n z^n}. \tag{40}$$

We have

$$|f(z)| = \frac{|z^m||a_0/z^m + a_1/z^{m-1} + \cdots + a_m|}{|z^n||b_0/z^n + b_1/z^{n-1} + \cdots + b_n|},$$

but as $|z|$ increases, terms such as $|c|/|z|^r$, and hence ones such as c/z^r, tend to zero, showing that when $|z|$ is large $|f(z)|$ can be overestimated by

$$|f(z)| \leq \frac{K}{|z|^{n-m}}, \tag{41}$$

for some finite positive constant K, and $n - m$ positive, zero or negative.

THEOREM 15.15

estimating the rate of decay of an integral on a circular arc as its radius $\to \infty$

Estimation of $\int_{\Gamma_R} f(z)dz$ when $f(z)$ decays rapidly for large $|z|$ Let $f(z)$ be analytic in the upper half of the complex plane with the exception of a finite number of poles at the points $z_1, z_2, \ldots, z_N$. Then if for $|z| > R$ the function $f(z)$ is such that $|f(z)| < K/|z|^{1+\delta}$, with K and δ positive constants,

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z)dz = 0,$$

where Γ_R is the part of the circle $|z| = R$ that lies in the upper half of the complex plane.

Proof On Γ_R we have $z = Re^{i\theta}$, so from the usual integral inequality,

$$\left|\int_{\Gamma_R} f(z)dz\right| = \left|\int_0^\pi f(Re^{i\theta})Rie^{i\theta}d\theta\right| \le \int_0^\pi |f(Re^{i\theta})Rie^{i\theta}|d\theta$$
$$< \int_0^\pi \left(\frac{K}{R^{1+\delta}}\right) Rd\theta = \frac{K\pi}{R^\delta}.$$

The result of the theorem now follows directly by taking the limit as $R \to \infty$. ■

Theorem 15.15 provides the justification for the use of property (39) that was assumed in Example 15.22, because for large $|z|$ it follows from (41) that a constant K can be found such that $|f(z)| < K/|z|^4$, showing that in this case $\delta = 3$.

EXAMPLE 15.28

Evaluate the integral

$$\int_0^\infty \frac{a+x^2}{1+x^4}dx \quad \text{where } a \text{ is a real constant.}$$

Solution The integrand is an even function of x, so

$$\int_0^\infty \frac{a+x^2}{1+x^4}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{a+x^2}{1+x^4}dx.$$

The function $f(z) = (a + z^2)/(1 + z^4)$ reduces to the required integrand on the real axis, so integrating $f(z)$ around the contour in Fig. 15.8a and using the residue theorem leads to the result

$$\int_\Gamma \frac{a+z^2}{1+z^4}dz = \int_{-R}^R \frac{a+x^2}{1+x^4}dx + \int_{\Gamma_R} \frac{a+z^2}{1+z^4}dz = 2\pi i\{\text{Res}[f(z), z_0] + \text{Res}[f(z), z_1]\},$$

when R is sufficiently large that Γ contains the two of the four simple poles of $f(z)$ that lie in the upper half of the complex plane at the points $z_0 = e^{\pi i/4}$ and $z_1 = e^{3\pi i/4}$. These poles occur at the same points as those of Example 15.27, though the residues are different.

We find that

$$\text{Res}[f(z), z_0] = \frac{a+i}{2\sqrt{2}(i-1)} \quad \text{and} \quad \text{Res}[f(z), z_1] = \frac{a-i}{2\sqrt{2}(1+i)},$$

so substituting these values in the preceding result gives

$$\int_{-R}^{R} \frac{a+x^2}{1+x^4}dx + \int_{\Gamma_R} \frac{a+z^2}{1+z^4}dz = 2\pi i\left\{\frac{a+i}{2\sqrt{2}(i-1)} + \frac{a-i}{2\sqrt{2}(1+i)}\right\} = \frac{\pi}{\sqrt{2}}(a+1).$$

Theorem 15.15 applies, because for large $|z|$ a positive constant K can be found such that $|f(z)| < K/|z|^2$ corresponding to $\delta = 1$, so proceeding to the limit as $R \to \infty$ gives

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{a+x^2}{1+x^4}dx = \frac{\pi}{\sqrt{2}}(a+1).$$

To justify removing the P.V. symbol we need to show that the improper integral is convergent. As the integrand $(a+x^2)/(1+x^4)$ is an even function of x and its integral over any finite interval is finite, it will be sufficient to show that $\int_R^\infty \frac{a+x^2}{1+x^4}dx$ is finite for any $R > 0$. This is indeed so, because for large R it is always possible to find an $M > 0$ such that $(a+x^2)/(1+x^4) \le M/x^2$, and $\int_R^\infty M/x^2 dx = M/R$ is finite, so we are justified in writing

$$\int_0^\infty \frac{a+x^2}{1+x^4}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{a+x^2}{1+x^4}dx = \frac{\pi}{2\sqrt{2}}(a+1).$$ ■

We now combine Theorem 15.5 and the residue theorem to arrive at the following theorem that enables the rapid evaluation of a certain type of improper integral.

THEOREM 15.16

a useful theorem when $|f(z)|$ decays rapidly as $|z| \to \infty$

Integration of functions that decay rapidly as $|z|$ becomes large Let $f(z)$ be analytic in the upper half of the complex plane with the exception of a finite number of poles at the points $z_1, z_2, \dots, z_N$, and let no poles of $f(z)$ lie on the real axis. Then if for $|z| > R$ the function $f(z)$ is such that $|f(z)| < K/|z|^{1+\delta}$, where K and δ are positive constants,

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^{N} \text{Res}[f(z), z_k].$$ ■

Notice that when the function $f(z)$ in Theorem 15.16 is a rational function of the form (40), the condition $|f(z)| < K/|z|^{1+\delta}$ when $|z| > R$ becomes the condition $n - m \ge 2$.

EXAMPLE 15.29

Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^4}dx.$$

Solution We set $f(z) = z^2/(1+z^2)^4$, because this reduces to the required integrand on the real axis, and notice that the conditions of Theorem 15.16 are satisfied by $f(z)$, because for large $|z|$ it behaves like $K/|z|^6$. Writing

$$f(z) = \frac{z^2}{(z-i)^4(z+i)^4},$$

shows that $f(z)$ only has a single pole of order 4 at $z = i$ in the upper half of the

complex plane with

$$\operatorname{Res}[f(z), i] = -\frac{i}{32}.$$

From Theorem 15.16 we have

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x^2 dx}{(1+x^2)^4} = 2\pi i \left(-\frac{i}{32}\right) = \frac{\pi}{16}.$$

The P.V. symbol can be omitted because the integrand is everywhere continuous and finite, and for large x the integrand behaves like $1/x^6$ showing that the improper integral is convergent, so we conclude that

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(1+x^2)^4} = \frac{\pi}{16}.$$

■

(c) Improper Integrals with Integrands of the Form $e^{imz}Q(z)$

Another important type of improper integral that occurs is one where the integrand is of the form $f(z) = e^{imz}Q(z)$, involving the product of an exponential factor e^{imz} with $m > 0$ and a rational function $Q(z)$. If the method of residues is to be used to evaluate improper integrals of this type it is necessary to find conditions that will ensure the validity of the limit in (39) when $f(z)$ is of this form.

The first step when seeking to establish such a condition is to prove a result known as the **Jordan inequality**, and an associated result that we will call the **Jordan integral inequality**.

LEMMA 15.1

the Jordan inequality and integral inequality

The Jordan inequality and integral inequality

(a) $\dfrac{2\theta}{\pi} \le \sin\theta \le \theta, \quad \text{for } 0 \le \theta \le \pi/2 \quad \text{(Jordan inequality)}$

(b) $\displaystyle\int_0^{\pi/2} e^{-k\sin\theta} d\theta \le \frac{\pi}{2k}(1 - e^{-k}), \quad \text{for } k > 0 \quad \text{(Jordan integral inequality).}$

Proof

(a) Assuming the inequality to be true, division by θ allows it to be written as

$$1 \ge \frac{\sin\theta}{\theta} \ge \frac{2}{\pi}, \quad \text{for } 0 \le \theta \le \pi/2.$$

Setting $S(\theta) = \sin\theta/\theta$ we have $S(\pi/2) = 2/\pi$, and from L'Hospital's rule

$$S(0) = \lim_{\theta\to 0} S(\theta) = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1,$$

so the upper and lower limits of the Jordan inequality have been established. The inequality will be proved if we can show that $S'(\theta) < 0$ for $0 \le \theta \le \pi/2$, because then $S(\theta)$ will be a strictly decreasing function of θ in the interval.

Differentiation of $S(\theta)$ gives

$$S'(\theta) = \frac{\theta\cos\theta - \sin\theta}{\theta^2},$$

so the sign of $S'(\theta)$ is determined by the sign of $h(\theta) = \theta\cos\theta - \sin\theta$. Using the results $h(0) = 0$ and $h'(\theta) = -\theta\sin\theta$ shows that $h'(\theta) \le 0$ for $0 \le \theta \le \pi/2$, so $h(\theta)$ and hence also $S(\theta)$ are strictly decreasing functions of θ in the given interval, and the Jordan inequality is proved.

(b) The integral form of the inequality follows by replacing $\sin\theta$ by $2\theta/\pi$ in the integrand $e^{-k\sin\theta}$ and then integrating to obtain the stated result. ■

We now use the Jordan integral inequality to prove the next result known as **Jordan's lemma**.

THEOREM 15.17

the useful Jordan's lemma

Jordan's lemma Let m be a positive constant and $Q(z)$ be a continuous function in the upper half of the complex plane, such that for $|z| \ge R_0$

$$M_R = \max_{z\in\Gamma_R} |Q(z)| \to 0 \quad \text{as } R \to \infty,$$

where Γ_R is the semicircle $|z| = R$ in the upper half of the complex plane. Then

$$\lim_{R\to\infty} \int_{\Gamma_R} e^{imz} Q(z)dz = 0.$$

Proof Let z lie on the semicircle Γ_R with $R > R_0$, so $z = Re^{i\theta}$ and $dz = i\,Re^{i\theta}d\theta$. Then

$$|e^{imz}| = |e^{imR(\cos\theta + i\sin\theta)}| = e^{-mR\sin\theta},$$

and on Γ_R we have

$$\left|\int_{\Gamma_R} e^{imz} Q(z)dz\right| \le \max_{z\in\Gamma_R} |Q(z)| \int_0^{\pi} e^{-mR\sin\theta} R d\theta = RM_R \int_0^{\pi} e^{-mR\sin\theta} d\theta.$$

The last integral cannot be evaluated as it stands, but because of the symmetry of $\sin\theta$ about the value $\theta = \pi/2$ the integral can be written as

$$RM_R \int_0^{\pi} e^{-mR\sin\theta} d\theta = 2RM_R \int_0^{\pi/2} e^{-mR\sin\theta} d\theta.$$

As the interval of integration is now $0 \le \theta \le \pi/2$, we can apply the Jordan integral inequality to arrive at the estimate

$$2RM_R \int_0^{\pi/2} e^{-mR\sin\theta} d\theta \le \frac{\pi M_R}{m}(1 - e^{-mR}).$$

Thus,

$$\left|\int_{\Gamma_R} e^{imz} Q(z)dz\right| \le \frac{\pi M_R}{m}(1 - e^{-mR}),$$

but by hypothesis $M_R \to 0$ as $R \to \infty$, so the right-hand side of this inequality vanishes and the result is proved. ■

Coupling Jordan's lemma with the residue theorem, we arrive at the following theorem, which enables the rapid evaluation of improper integrals with integrands that involve a product of an exponential factor and a rational function.

THEOREM 15.18

Integrals with integrands of the form $e^{imz}Q(z)$

Integration of functions of the form $e^{imz}Q(z)$ Let $m > 0$ be a real constant, and $f(z) = e^{imz}Q(z)$ be analytic in the upper half of the complex plane with the exception of a finite number of poles at the points $z_1, z_2, \ldots, z_N$, and let no poles of $f(z)$ lie on the real axis. Then if for $|z| > R$ the function $Q(z)$ is such that for all z in the upper half of the complex plane

$$\lim_{|z|\to\infty} |Q(z)| \to 0,$$

it follows that

$$\text{P.V.} \int_{-\infty}^{\infty} e^{imx} Q(x)dx = 2\pi i \sum_{k=1}^{N} \text{Res}[e^{imz}Q(z), z_k].$$

■

The following theorem is often useful in establishing the convergence of integrals obtained by using Theorem 15.19, and so justifying the omission of the P.V. symbol.

THEOREM 15.19

Convergence of integrals with integrands of the form $e^{imz}Q(z)$ Let $Q(x) > 0$ be a strictly decreasing function of x for $0 \le x < \infty$ such that $\lim_{x\to\infty} Q(x) = 0$. Then, provided the integrands are finite at the origin, the improper integrals $\int_0^\infty Q(x)\cos mx dx$ and $\int_0^\infty Q(x)\sin mx dx$ are convergent. Furthermore, if $Q(x)$ is an even function, the improper integral $\int_{-\infty}^{\infty} Q(x)\cos mx dx$ is convergent, and if $Q(x)$ is an odd function the improper integral $\int_{-\infty}^{\infty} Q(x)\sin mx dx$ is convergent.

Proof As $Q(x) > 0$, the sign of the integrand in $\int_0^\infty Q(x)\cos mx dx$ will be determined by the sign of $\cos mx$. The function $\cos mx$ changes sign in adjacent intervals of the form $(2n-1)\pi/2m < x < (2n+1)\pi/2m$, for $n = 1, 2, \ldots$, so setting

$$I_n = \int_{(2n-1)\pi/2m}^{(2n+1)\pi/2m} Q(x)|\cos mx|dx$$

allows us to write

$$\int_{(2n-1)\pi/2m}^{(2n+1)\pi/2m} Q(x)\cos mx dx = (-1)^n I_n.$$

This result enables the original integral to be written as

$$\int_0^\infty Q(x)\cos mx dx = \int_0^{\pi/2m} Q(x)\cos mx dx + \sum_{n=1}^{\infty} (-1)^n I_n.$$

By hypothesis, $Q(x)$ is a strictly decreasing function of x, so $0 < I_{n+1} < I_n$, but $\lim_{x\to\infty} Q(x) = 0$, so we also have $\lim_{x\to\infty} I_n = 0$. The series $\sum_{n=1}^{\infty}(-1)^n I_n$ is seen to be an *alternating series* satisfying the alternating series test for convergence, and so has a finite sum. As the integrand is assumed to be finite at the origin, the term $\int_0^{\pi/2m} Q(x)\cos mx dx$ is finite, showing that the integral $\int_0^\infty Q(x)\cos mx dx$ has a finite sum. This has proved the integral to be convergent, thus allowing the P.V. symbol to be omitted. The convergence of $\int_0^\infty Q(x)\sin mx dx$ can be established in similar fashion. In the case of integrals over an infinite interval, the conditions

imposed on $Q(x)$ in the last part of the theorem allow the integrals to be reduced to one of the cases just considered, so the proof is complete. ■

EXAMPLE 15.30 Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)^4} dx.$$

Solution The real part of $f(z) = \exp(iz)/(1+z^2)^4$ reduces to the required integrand on the real axis, so we take this for our integrand. An attempt to use the more obvious choice of integrand $\cos z/(1+z^2)^4$ must be avoided because it would introduce unnecessary complications due to the behavior of $f(z)$ as $|z| \to \infty$. As in Example 15.27, the integrand only has a single pole of order 4 located at $z = i$ in the upper half of the complex plane. A routine calculation shows that the residue at $z = i$ is

$$\text{Res}[f(z), i] = -\frac{37i}{96e}.$$

The conditions of Theorem 15.19 are seen to be satisfied, so it follows that

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\exp(iz)}{(1+x^2)^4} dx = 2\pi i \left(-\frac{37i}{96e}\right) = \frac{37\pi}{48}.$$

Equating the real parts of the expressions on each side of the equation gives

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)^4} dx = \frac{37\pi}{48}.$$

The justification for the removal of the P.V. symbol follows from the form of proof used in Theorem 15.19 by setting

$$I_n = \int_{(2n+1)\pi/2}^{(2n+3)\pi/2} \frac{|\cos x|}{(1+x^2)^4} dx \quad \text{with } n = 1, 2, \ldots.$$

Consequently, we can write

$$\int_{0}^{\infty} \frac{\cos x}{(1+x^2)^4} dx = \frac{37\pi}{96} \quad \text{or, equivalently,} \quad \int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)^4} dx = \frac{37\pi}{48}.$$

Had the imaginary parts been equated, we would have obtained the result

$$\int_{-\infty}^{\infty} \frac{\sin x}{(1+x^2)^4} dx = 0,$$

which is to be expected because the integral is convergent and the integrand is an *odd* function. ■

EXAMPLE 15.31 Evaluate the integral

$$\int_{0}^{\infty} \frac{x \sin x}{(x^2+1)^2} dx.$$

Solution The integrand is an even function of x, so we will consider the integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+1)^2} dx.$$

We integrate the function $f(z) = z\exp(iz)/(z^2+1)^2$ around the contour Γ in Fig. 15.8a and notice that when $|z|$ is sufficiently large, $f(z)$ only has a single pole of order 2 at the point $z = i$ inside Γ. We find that

$$\mathrm{Res}[f(z), i] = \frac{1}{4e},$$

so as $f(z)$ satisfies the conditions of Theorem 15.18, after equating the imaginary parts we have

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{x\exp(ix)}{(x^2+1)^2}dx = 2\pi i\left(\frac{1}{4e}\right) = \frac{\pi i}{2e}$$

and so

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{x\sin x}{(x^2+1)^2}dx = \frac{\pi}{2e}.$$

The conditions of Theorem 15.19 are satisfied if in its proof we define

$$I_n = \int_{n\pi}^{(n+1)\pi} \frac{x|\sin x|}{(1+x^2)^2}dx \quad \text{for } n = 1, 2, \ldots,$$

so the P.V. symbol can be omitted, leading to the result

$$\int_0^{\infty} \frac{x\sin x}{(x^2+1)^2}dx = \frac{\pi}{4e}.$$ ■

The last example is somewhat different, because it involves an integrand that is an entire function, so by the Cauchy–Goursat theorem its integral around any simple closed contour must be zero, though in this case the contour used is a sector of a circle and not a semicircle.

EXAMPLE 15.32 By considering the integral

$$\int_\Gamma \exp(iz^2)dz$$

around a suitable contour Γ, show that

$$\int_0^{\infty} \cos x^2 dx = \int_0^{\infty} \sin x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

Fresnel integrals

Solution These integrals, called the **Fresnel integrals**, are of importance in engineering and physics in connection with the study of diffraction phenomena. For reasons that will appear later, we take for the positively oriented contour Γ the boundary of the sector of a circle shown in Fig. 15.9 with the internal angle $\pi/4$, and the positively directed circular arc AB of radius R denoted by Γ_R.

The integrand $\exp(iz^2)$ is an entire function, so from the Cauchy–Goursat theorem

$$\int_\Gamma \exp(iz^2)dz = 0.$$

To derive the required improper integrals, we represent the integral around Γ as the sum of integrals along the real axis from O to A, along the arc Γ_R from A to B, and along the radial line from B to O (take note of the direction of integration

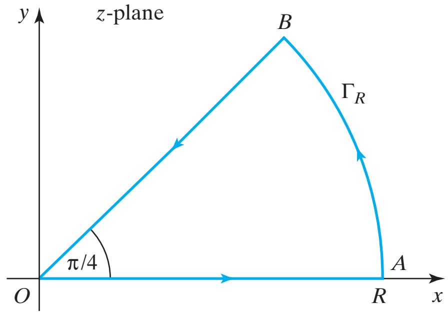


FIGURE 15.9 The sector bounded by the contour Γ.

along this line), as a result of which we find that

$$\int_\Gamma \exp(iz^2)dz = \int_{OA} \exp(iz^2)dz + \int_{\Gamma_R} \exp(iz^2)dz + \int_{BO} \exp(iz^2)dz = 0.$$

Line segment AB lies on the real axis, so on AB we have $z = x$ and hence $dz = dx$, whereas on the radial line OB inclined at an angle $\pi/4$ to the real axis $z = re^{i\pi/4}$, so here $dz = e^{i\pi/4}dr$ and $iz^2 = -r^2$. Using these results in the preceding equation reduces it to

$$\int_0^R \exp(ix^2)dx + \int_{\Gamma_R} \exp(iz^2)dz + e^{i\pi/4}\int_R^0 \exp(-r^2)dr = 0.$$

Reversing the limits of integration in the last integral and rearranging terms gives

$$\int_0^R \exp(ix^2)dx + \int_{\Gamma_R} \exp(iz^2)dz = e^{i\pi/4}\int_0^R \exp(-r^2)dr.$$

Taking the limit of this result as $R \to \infty$ gives

$$\text{P.V.} \int_0^\infty \exp(ix^2)dx + \lim_{R\to\infty} \int_{\Gamma_R} \exp(iz^2)dz = e^{i\pi/4}\frac{\sqrt{\pi}}{2},$$

where we have used the standard result from calculus that

$$\int_0^\infty \exp(-r^2)dr = \frac{1}{2}\sqrt{\pi}.$$

As neither Theorem 15.16 nor Theorem 15.18 apply to the integral around Γ_R, to make further progress we need to examine the limit

$$\lim_{R\to\infty} I_R = \lim_{R\to\infty} \int_{\Gamma_R} \exp(iz^2)dz.$$

On Γ_R we have $z = Re^{i\theta}$, with $0 \le \theta \le \pi/4$, so

$$\exp(iz^2) = \exp(iR^2\cos 2\theta)\cdot\exp(-R^2\sin 2\theta) \quad \text{and} \quad dz = iRe^{i\theta}d\theta,$$

showing that

$$I_R = \int_0^{\pi/4} \exp(iR^2\cos\theta)\cdot\exp(-R^2\sin 2\theta)iRe^{i\theta}d\theta.$$

To estimate this integral we take its modulus and use the standard integral inequality

$$|I_R| \le \int_0^{\pi/4} |\exp(iR^2\cos\theta)\exp(-R^2\sin 2\theta)iRe^{i\theta}|d\theta,$$

together with the fact that $|iRe^{i\pi/4}| = R$ and $|\exp(iR^2\cos 2\theta)| = 1$, to arrive at the inequality

$$|I_R| \le R\int_0^{\pi/4} \exp(-R^2\sin 2\theta)d\theta.$$

The integral on the right cannot be evaluated in terms of simple functions, but it can be estimated with the help of the Jordan inequality. The interval of integration involved is $0 \le \theta \le \pi/4$, so on this interval $0 \le 2\theta \le \pi/2$. If we replace θ by 2θ in the Jordan inequality, the result becomes

$$\sin 2\theta \le \frac{4\theta}{\pi},$$

from which we see that

$$\exp(-R^2\sin 2\theta) \le \exp\left(-\frac{4R^2\theta}{\pi}\right),$$

leading to the inequality

$$|I_R| \le R\int_0^{\pi/4} \exp\left(-\frac{4R^2\theta}{\pi}\right)d\theta = \frac{\pi}{4R}[1-\exp(-R^2)].$$

Taking the limit of this last result as $R \to \infty$ gives $\lim_{R\to\infty}|I_R| = 0$, showing that the integral around the arc Γ_R vanishes in the limit as $R \to \infty$.

Using this result in the contour integral around Γ, we conclude that

$$\text{P.V.}\int_0^{\infty} \exp(ix^2)dx = e^{i\pi/4}\frac{\sqrt{\pi}}{2}.$$

The Fresnel integrals follow by omitting the P.V. symbol and equating the respective real and imaginary parts on each side of this equation to obtain

$$\int_0^{\infty} \cos x^2 dx = \int_0^{\infty} \sin x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

The justification for the removal of the P.V. symbols follows by using an argument similar to the one employed in Theorem 15.19, because as x increases the integrands oscillate more frequently, causing integrals over successive periods to form convergent alternating series.

The reason for choosing the contour Γ to be the boundary of a sector with angle $\pi/4$ is now apparent, because were the angle to exceed $\pi/4$, Jordan's inequality could not be used to estimate $|I_R|$. If, on the other hand, the angle were to be less than $\pi/4$ the form of the resulting integrals would be different, and to evaluate them the values of the Fresnel integrals would need to be known. ■

(d) Improper Integrals with Poles on the Real Axis

We now consider improper integrals where a simple pole of the integrand occurs on the real axis. Let $f(z)$ have a simple pole located at a point x_0 on the real axis

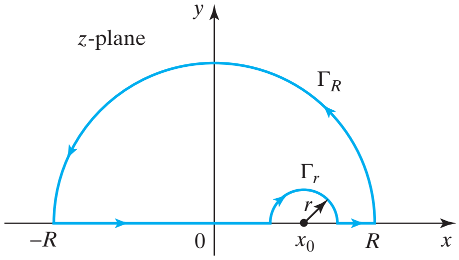


FIGURE 15.10 An indentation Γ_r at x_0 on the real axis.

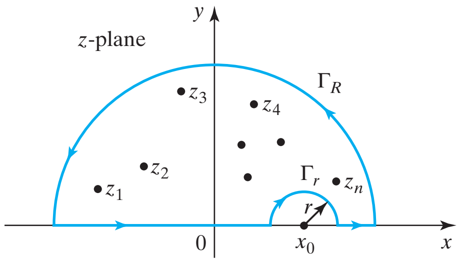


FIGURE 15.11 A contour Γ indented at x_0 on the real axis.

that forms part of the integration path in a contour integral. To prevent the contour passing through the pole, the contour is deformed in a neighborhood of x_0 by a small semicircle of radius r centered on x_0 extending into the upper half of the complex plane, as shown in Fig. 15.10, and we denote this **indentation** by Γ_r.

indentations

The Laurent series representation of $f(z)$ at x_0 is

$$f(z) = \frac{a_{-1}}{z - x_0} + \sum_{n=0}^{\infty} a_n (z - x_0)^n,$$

where $a_{-1} = \text{Res}[f(z), x_0]$. On Γ_r, $z = x_0 + re^{i\theta}$ and $dz = ire^{i\theta}d\theta$ with $0 \le \theta \le \pi$, so integrating around Γ_r in the positive sense gives

$$\begin{aligned}\lim_{r\to 0}\int_{\Gamma_r} f(z)dz &= \lim_{r\to 0}\int_0^{\pi} \frac{a_{-1}}{re^{i\theta}} ire^{i\theta}d\theta + \lim_{r\to 0}\sum_{n=0}^{\infty} a_n \int_0^{\pi} r^n e^{in\theta} ire^{i\theta}d\theta \\ &= ia_{-1}\int_0^{\pi} d\theta = i\pi a_{-1} \\ &= i\pi\,\text{Res}[f(z), x_0].\end{aligned}$$

So, in the limit as $r \to 0$, we have shown that integrating in the positive sense around the semicircular indentation Γ_r above the simple pole located at the point x_0 on the real axis yields $\pi i\,\text{Res}[f(z), x_0]$. This result is seen to be *half* the result that would have been obtained had the integration been taken around a circle with the pole at x_0 at its center.

integration around an indented simple pole

This same form of argument establishes the more general result that if a simple pole is located at z_0, then integration around the pole using a path in the form of a sector of a circle Γ_r, located at z_0 with an arbitrarily small radius r and an internal angle α yields the result

$$\int_{\Gamma_r} f(z)dz = i\alpha\,\text{Res}[f(z), z_0]. \tag{42}$$

Consider a function $f(z)$ that has a finite number of poles located at z_1, $z_2, \ldots, z_n$ in the upper half of the complex plane and a simple pole on the real axis at x_0. Let the positively oriented contour Γ be the one shown in Fig. 15.11, where the indentation above the pole at x_0 is denoted by Γ_r, and Γ_R denoting the

semicircle of radius R. Then, when R is sufficiently large that all of the poles above the real axis lie inside Γ, integrating around Γ in the positive sense gives

$$\int_{\Gamma} f(z)dz = \int_{-R}^{x_0 - r} f(x)dx + \int_{\Gamma_r} f(z)dz + \int_{x_0 + r}^{R} f(x)dx + \int_{\Gamma_R} f(z)dz$$
$$= 2\pi i \sum_{k=1}^{n} \text{Res}[f(z), z_k] \quad \text{when} \quad \lim_{R\to\infty} \int_{\Gamma_R} f(z)dz = 0.$$

Before proceeding to the limit as $R \to \infty$ and $r \to 0$, we notice that the integration around Γ_r, corresponding to $\alpha = \pi$ in (42), is in the *negative* sense, so after the limits have been taken, the result becomes

$$\int_{-\infty}^{x_0-} f(x)dx - \pi i \text{Res}[f(z), x_0] + \int_{x_0+}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^{n} \text{Res}[f(z), z_k].$$

Combining the integrals and rearranging terms gives

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = \pi i \text{Res}[f(z), x_0] + 2\pi i \sum_{k=1}^{n} \text{Res}[f(z), z_k]. \tag{43}$$

This result extends immediately to a function with m simple poles located on the real axis and so leads to the following theorem.

THEOREM 15.20

integrals involving functions with poles on the real axis

The residue theorem when poles are located on the real axis Let an analytic function $f(z)$ have n poles at the points $z_1, z_2, \ldots, z_n$ in the upper half of the complex plane and m simple poles at the points $x_1, x_2, \ldots, x_m$ on the real axis. Then, provided $\lim_{R\to\infty} \int_{\Gamma_R} f(z)dz = 0$ where Γ_R is the semicircle $|z| = R$ in the upper half of the complex plane,

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = \pi i \sum_{k=1}^{m} \text{Res}[f(z), x_k] + 2\pi i \sum_{k=1}^{n} \text{Res}[f(z), z_k].$$ ■

EXAMPLE 15.33

Evaluate the integral

$$\int_0^{\infty} \frac{\sin x}{x} dx.$$

Solution The integrand is an even function of x, and because $\lim_{x\to 0}(\sin x/x) = 1$ the singularity at the origin is removable, so we consider the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx.$$

To evaluate this integral we integrate the function $f(z) = \exp(iz)/z$ around a contour Γ indented at the origin, as shown in Fig. 15.12, because using the function $f(z) = \sin z/z$ would introduce unnecessary complications when z is large.

The only pole of $f(z)$ is a simple pole at the origin, where $\text{Res}[f(z), 0] = 1$, so as the conditions of Jordan's lemma are satisfied, we can use Theorem 15.20 to evaluate the integral. An application of the theorem gives

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\exp(ix)}{x} dx = \pi i.$$

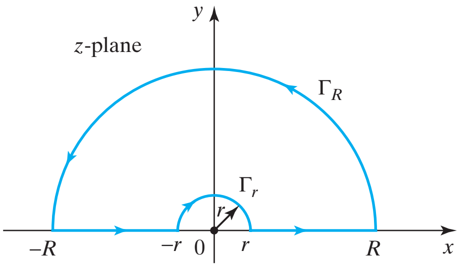


FIGURE 15.12 The contour Γ indented at the origin.

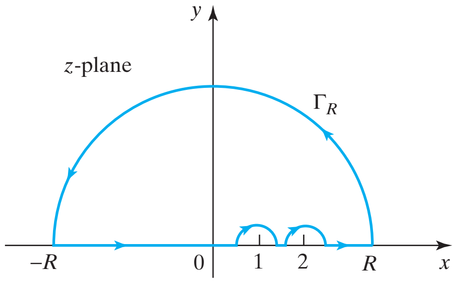


FIGURE 15.13 The contour Γ indented at $x = 1$ and $x = 2$ on the real axis.

Equating the imaginary parts of the expressions on each side of the last equation gives

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi.$$

As $x = 0$ is a removable singularity the integrand $\sin x/x$ is finite at the origin, so this fact together with the form of argument used in Example 15.30 justifies the removal of the P.V. symbol, and we have proved that

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}.$$ ■

EXAMPLE 15.34 Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)(x^2-3x+2)} dx.$$

Solution We choose for the integrand the function $f(z) = \exp(iz)/[(z^2+1)(z^2 - 3z+2)]$. This has simple poles at $z = \pm i$, $z = 1$, and $z = 2$. Modifying the contour in Fig. 15.10 to allow for the two simple poles on the real axis leads to integration around the indented contour shown in Fig. 15.13, which contains the simple pole at $z = i$. The usual calculations show that

$$\text{Res}[f(z), i] = \frac{(3-i)}{20e}, \quad \text{Res}[f(z), 1] = -\frac{1}{2}(\cos(1) + i\sin(1)) \quad \text{and}$$

$$\text{Res}[f(z), 2] = \frac{1}{5}(\cos(2) + i\sin(2)).$$

The conditions of Theorem 15.20 are seen to be satisfied, so

$$\begin{aligned} &\text{P.V.} \int_{-\infty}^{\infty} \frac{\exp(ix)}{(x^2+1)(x^2-3x+2)} dx \\ &= 2\pi i \text{Res}[f(z), i] + \pi i\{\text{Res}[f(z), 1] + \text{Res}[f(z), 2]\} \\ &= 2\pi i\left(\frac{3-i}{20e}\right) + \pi i\left(-\frac{[\cos(1) + i\sin(1)]}{2}\right) + \pi i\left(\frac{\cos(2) + i\sin(2)}{5}\right). \end{aligned}$$

Equating the real parts on each side of this equation shows that

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)(x^2-3x+2)} dx = \frac{\pi}{10}\left(\frac{1}{e} + 5\sin(1) - 2\sin(2)\right).$$

In this case, because of the complexity of the integrand, no attempt will be made to investigate whether the P.V. symbol can be omitted.

Although not required, equating imaginary parts on each side of the equation shows that

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\sin x}{(x^2+1)(x^2-3x+2)} dx = \frac{\pi}{10}\left(\frac{3}{e} + 2\cos(2) - 5\cos(1)\right).$$

This determination of two real improper integrals when only one was required is typical of the evaluation of real integrals by contour integration. ■

(e) Improper Integrals with Branch Points

Finally, we consider improper integrals of functions with a branch point. To evaluate these by means of contour integration it is necessary to cut the complex plane in an appropriate manner to make the integrand single valued, and to specify the branch of the integrand that is to be used. An important class of integrals of this type are of the form

$$\int_0^{\infty} x^{\alpha-1} P(x) dx, \tag{44}$$

where α is not an integer and $P(x)$ is a rational function of x. This integral will have a finite value if $P(x)$ has no poles on the positive real axis and it is such that

$$\lim_{z\to 0} |z|^{\alpha} P(z) = 0 \quad \text{and} \quad \lim_{|z|\to\infty} |z|^{\alpha} P(z) = 0. \tag{45}$$

Provided $z = 0$ is neither a pole nor a zero of $P(z)$, the first of these conditions implies that $\alpha > 0$. Let the rational function $P(z)$ with real coefficients $a_0, a_1, \ldots, a_m$ and $b_0, b_1, \ldots, b_n$ be written

$$P(z) = \frac{a_0 z^m + a_1 z^{m-1} + \cdots + a_m}{b_0 z^n + b_1 z^{n-1} + \cdots + b_n},$$

so that for large $|z|$ a constant K exists such that $P(z) < K/|z^{n-m}|$. Then the second condition in (45) will be satisfied when $n - m - \alpha > 0$. Taken together, these conditions show the integral will have a finite value when $0 < \alpha < n - m$, and they also imply that

$$\lim_{|z|\to\infty} |P(z)| = 0.$$

To take account of the fact that $z^{\alpha-1}$ is many valued and has a branch point at the origin, it is necessary to cut the complex plane to make $z^{\alpha-1}$ (and hence the integrand) single valued, and then to choose a branch of $z^{\alpha-1}$. The cut we will make is along the positive real axis up to and including the origin, so that arg $z = \theta + 2k\pi$, with $k = 0, \pm 1, \pm 2, \ldots$, and θ in the interval $0 \le \theta \le 2\pi$. The contour Γ that will be used is shown in Fig. 15.14 and comprises the circular contour Γ_R with equation $|z| = R$, the cut with its sides immediately above and below the positive real axis, and the circular contour Γ_ρ with equation $|z| = \rho$ around the branch point at the origin. We will work with the branch corresponding to $k = 0$, so $z = re^{i\theta}$ and $z^{\alpha-1} = r^{\alpha-1}e^{(\alpha-1)\theta i}$. The principal branch is positive on the side of the cut that lies

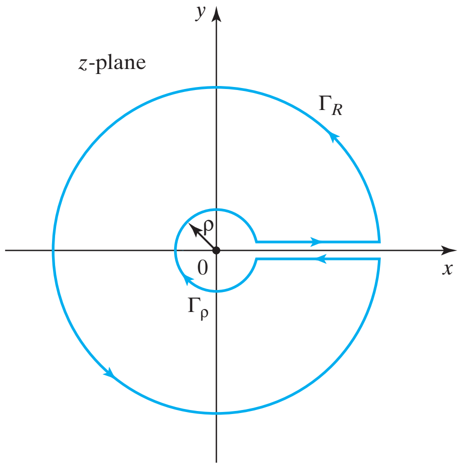


FIGURE 15.14 The contour Γ used to evaluate $\int_0^\infty x^{\alpha-1}P(x)dx$.

above the positive real axis. This branch of the function $z^{\alpha-1}P(z)$ is now single valued in the cut plane, so we can use the residue theorem to evaluate the integral.

When substituting for z in the various integrals that arise while integrating around Γ, it is necessary to express z in its modulus–argument form to take account of the different forms taken by the integrand $z^{\alpha-1}P(z)$ on either side of the cut. Setting $z = re^{i\theta}$, with $0 \le \theta \le 2\pi$, it follows that on AB $z = re^{0i} = r$ and $dz = dr$, so that

$$z^{\alpha-1}P(z) = r^{\alpha-1}P(r),$$

while on CD $z = re^{2\pi i}$ and $dz = e^{2\pi i}dr$, so then

$$z^{\alpha-1}P(z) = r^{\alpha-1}e^{(\alpha-1)2\pi i}P(r).$$

We now set $f(z) = z^{\alpha-1}P(z)$, and consider the case where $f(z)$ has poles at $z_1, z_2, \ldots, z_n$, none of which lies on the positive real axis. Integrating around the contour Γ in Fig. 15.14 gives

$$\int_\rho^R r^{\alpha-1}P(r)dr + \int_{\Gamma_R} z^{\alpha-1}P(z)dz + \int_R^\rho r^{\alpha-1}\exp[(\alpha-1)2\pi i]P(r)e^{2\pi i}dr$$
$$+ \int_{\Gamma_\rho} z^{\alpha-1}P(z)dz = 2\pi i\sum_{k=1}^{n}\mathrm{Res}[f(z), z_k].$$

The conditions (45) with $0 < \alpha < n - m$ ensure the vanishing of both the integral around Γ_R in the limit as $R \to \infty$ and the integral around Γ_ρ as $\rho \to 0$, so taking the limit as $R \to \infty$ and $\rho \to 0$ reduces the preceding result to

$$\int_0^\infty r^{\alpha-1}P(r)dr + e^{2\pi i\alpha}\int_\infty^0 r^{\alpha-1}P(r)dr = 2\pi i\sum_{k=1}^{n}\mathrm{Res}[f(z), z_k].$$

integration around a branch point

Replacing the dummy variable r by x and rearranging terms, we arrive at the general result

$$\int_0^\infty x^{\alpha-1} P(x)dx = \frac{2\pi i}{1-e^{2\pi i\alpha}} \sum_{k=1}^{n} \mathrm{Res}[f(z), z_k]. \tag{46}$$

This result forms our next theorem.

THEOREM 15.21

Evaluation of integrals of the form $\int_0^\infty x^{\alpha-1}P(x)dx$ Let $f(z) = z^{\alpha-1}P(z)$ with α not an integer and

$$P(z) = \frac{a_0 z^m + a_1 z^{m-1} + \cdots + a_m}{b_0 z^n + b_1 z^{n-1} + \cdots + b_n},$$

where the coefficients $a_0, a_1, \ldots, a_m$ and $b_0, b_1, \ldots, b_n$ are all real, $0 < \alpha < n-m$, and $P(z)$ has neither a pole nor a zero at the origin. In addition, let the poles of $P(z)$ located at $z_1, z_2, \ldots, z_n$ be such that none lies on the positive real axis. Then

$$\int_0^\infty x^{\alpha-1} P(x)dx = \frac{2\pi i}{1-e^{2\pi i\alpha}} \sum_{k=1}^{n} \mathrm{Res}[f(z), z_k]. \qquad \blacksquare$$

EXAMPLE 15.35

Find a condition on α that ensures that the integral

$$\int_0^\infty \frac{x^{\alpha-1}}{x^2+1}dx$$

exists, and evaluate the integral subject to this condition.

Solution In the notation of Theorem 15.21, the rational function $P(z) = 1/(1+z^2)$, so $m = 0$ and $n = 2$. The condition on α that ensures the existence of the integral is $0 < \alpha < n - m$, so we must have $0 < \alpha < 2$. The function $P(z)$ has simple poles at $z = \pm i$, neither of which lies on the positive real axis, and $P(0) \neq 0$, so all the conditions of Theorem 15.21 are satisfied.

Using the result of the theorem with $f(z) = z^{\alpha-1}/(1+z^2)$ we find that

$$\mathrm{Res}[f(z), i] = \lim_{z\to i}\left[(z-i)\frac{z^{\alpha-1}}{(z-i)(z+i)}\right] = \lim_{z\to i}\left[\frac{z^{\alpha-1}}{z+i}\right] = \frac{i^{\alpha-1}}{2i} = \frac{i^{\alpha-2}}{2},$$

but $i = e^{\pi i/2}$, so

$$\mathrm{Res}[f(z), i] = \frac{1}{2}e^{(\alpha-2)\pi i/2} = -\frac{1}{2}e^{\alpha\pi i/2}.$$

Similarly,

$$\mathrm{Res}[f(z), -i] = \lim_{z\to -i}\left[(z+i)\frac{z^{\alpha-1}}{(z-i)(z+i)}\right] = \lim_{z\to -i}\left[\frac{z^{\alpha-1}}{z-i}\right] = \frac{(-i)^{\alpha-1}}{-2i} = \frac{(-i)^{\alpha-2}}{2},$$

but $-i = e^{3\pi i/2}$, so

$$\mathrm{Res}[f(z), -i] = \frac{1}{2}e^{(\alpha-2)3\pi i/2} = -\frac{1}{2}e^{3\alpha\pi i/2}.$$

Using these residues in Theorem 15.21 gives

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x^2}dx = \frac{2\pi i}{1-e^{2\alpha\pi i}}\left[-\frac{e^{\alpha\pi i/2}}{2} - \frac{e^{3\alpha\pi i/2}}{2}\right] = \pi i\left[\frac{e^{\alpha\pi i/2}+e^{-\alpha\pi i/2}}{e^{\alpha\pi i}-e^{-\alpha\pi i}}\right]$$
$$= \frac{\pi\cos(\alpha\pi/2)}{\sin(\alpha\pi)},$$

and we have shown that

$$\int_0^\infty \frac{x^{\alpha-1}}{x^2+1}dx = \pi\frac{\cos(\alpha\pi/2)}{\sin(\alpha\pi)}, \qquad \text{when } \alpha \text{ is not an integer and with } 0<\alpha<2.$$

Different types of function with branch points can be evaluated by means of contour integration, provided the complex plane is cut in a suitable manner to make the integrand single valued and a branch of the function is specified. The integrand in the next example involves the logarithmic function that has a branch point at the origin and infinitely many branches.

EXAMPLE 15.36 Show that

$$\int_0^\infty \frac{\log x}{x^2+a^2}dx = \frac{\pi}{2a}\ln a \quad (a>0).$$

Solution The function $\log z$ has infinitely many branches, so we will work with the principal branch $\operatorname{Log} z$. The contour Γ to be used is shown in Fig. 15.15, in which the cut is made along the negative real axis, and an indentation is made around the branch point of $\operatorname{Log} z$ located at the origin. The contour Γ_R is the semicircle with the equation $|z| = R$ and $\operatorname{Im} z > 0$, and the contour Γ_ρ is the semicircle with the equation $|z| = \rho$ and $\operatorname{Im} z > 0$.

With the cut as shown in Fig. 15.15, $\operatorname{Arg} z = \theta$ is restricted to the interval $0 \le \theta \le \pi$, so $z = re^{i\theta}$ and $\operatorname{Log} z = \ln r + i\theta$. Setting $f(z) = \operatorname{Log} z/(z^2+a^2)$, we see that when R is large the only singularity of $f(z)$ inside the contour Γ is a simple pole at $z = ia$, where

$$\operatorname{Res}[f(z), ia] = \lim_{z\to ia}[(z-ia)f(z)] = \operatorname{Log}(ia)/(2ia),$$

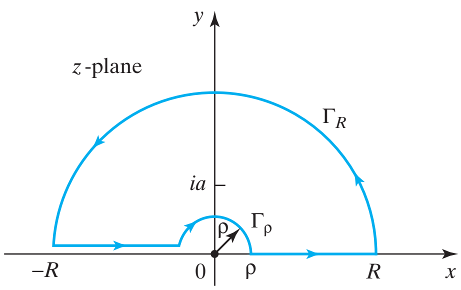


FIGURE 15.15 The contour Γ used to evaluate $\int_0^\infty \frac{\log x}{x^2+a^2}dx$.

but $i = e^{i\pi/2}$ so

$$\text{Res}[f(z), ia] = \frac{\ln a + i\pi/2}{2ia}.$$

On the positive real axis $z = re^{i0} = r$ and $dz = dr$, whereas on the negative real axis $z = re^{i\pi}$ and $dz = e^{i\pi}dr$, so as the simple pole at $z = ia$ lies inside Γ, integration around Γ leads to the result

$$\int_\rho^R \frac{\ln r}{r^2 + a^2} dr + \int_{\Gamma_R} f(z)dz + \int_R^\rho \frac{\ln r + i\pi}{r^2 e^{2\pi i} + a^2} e^{i\pi} dr + \int_{\Gamma_\rho} f(z)dz = 2\pi i \ \text{Res}[f(z), ia].$$

On Γ_R $z = Re^{i\theta}$ and $dz = i\,Re^{i\theta}d\theta$, so

$$\left|\int_{\Gamma_R} f(z)dz\right| = \left|\int_0^\pi \frac{\ln R + i\pi}{(R^2e^{2i\theta} + a^2)} iR \cdot e^{i\theta}d\theta\right|$$

$$\leq \int_0^\pi \frac{R \ln R}{|R^2 e^{2i\theta} + a^2|} d\theta \leq \int_0^\pi R \frac{\ln R}{(R^2 - a^2)} d\theta \leq \pi \left(\frac{\ln R}{R}\right),$$

but as $\lim_{R\to\infty}(\ln R/R) = 0$ it follows from this that $\int_{\Gamma_R} f(z)dz \to 0$ as $R \to \infty$.

A similar argument shows $\int_{\Gamma_\rho} f(z)dz \to 0$ as $\rho \to 0$, because when ρ is small the integrand is approximated by the function $\rho \ln \rho$ that vanishes in the limit as $\rho \to 0$. Taking the limit at $R \to \infty$ and $\rho \to 0$, and using the factor $e^{i\pi} = -1$ to reverse the limits in the third integral on the left gives

$$\int_0^\infty \frac{\ln r}{r^2 + a^2} dr + \int_0^\infty \frac{\ln r + i\pi}{r^2 + a^2} dr = 2\pi i \left(\frac{\ln a + i\pi/2}{2ia}\right).$$

Equating the real parts on either side of the equation and replacing the dummy variable r by x gives the required result,

$$\int_0^\infty \frac{\log x}{x^2 + a^2} dx = \frac{\pi \ln a}{2a} \quad (a > 0).$$

Equating the imaginary parts and again replacing the dummy variable r by x gives the elementary result,

$$\int_0^\infty \frac{dx}{x^2 + a^2} = \frac{\pi}{2a}.$$ ■

Alternative accounts and more information about Taylor and Laurent series, residues, the evaluation of real integrals by means of contour integrals, and the treatment of contour integrals involving branch points can be found in references [6.1] to [6.4] and [6.6] to [6.9].

Summary

After reviewing the concept of the Cauchy principal value of a definite integral, the residue theorem was used to evaluate real integrals in terms of the limit of associated contour integrals as the contour becomes arbitrarily large. The cases considered involved integrands with poles strictly inside the contour of integration, part of which was along the real axis, integrands with poles both inside and on an indented contour, and integration around an integrand with a branch point.

EXERCISES 15.5

Integrands without poles on the real axis

In Exercises 1 through 6 evaluate the integrals using the contour in Fig. 15.8a.

1. $\displaystyle\int_0^\infty \frac{x^2}{(x^2+a^2)^2}\,dx \quad (a>0).$

2. $\displaystyle\int_{-\infty}^\infty \frac{x^2}{x^4+a^4}\,dx \quad (a>0).$

3. $\displaystyle\int_0^\infty \frac{x^2}{x^4+1}\,dx \quad (a>0).$

4. $\displaystyle\int_{-\infty}^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx$ $(a, b > 0).$

5. $\displaystyle\int_{-\infty}^\infty \frac{x^2}{(x^2+1)^2(x^2+4)}\,dx.$

6. $\displaystyle\int_{-\infty}^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)^2}\,dx$ $(a, b > 0, a \neq b).$

Integrands of the form $e^{imz}Q(z)$

In Exercises 7 through 11 evaluate the integrals using the contour in Fig. 15.8a.

7. $\displaystyle\int_0^\infty \frac{\cos x}{(x^2+a^2)^2}\,dx \quad (a>0).$

8. $\displaystyle\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}\,dx \quad (a, b>0).$

9. $\displaystyle\int_0^\infty \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx \quad (a, b>0).$

10. $\displaystyle\int_0^\infty \frac{x\sin x}{x^2+4}\,dx.$

11. $\displaystyle\int_0^\infty \frac{x^3\sin mx}{x^4+a^4}\,dx \quad (a>0).$

12. By integrating around the contour in Fig. 15.16, show that

$$\int_{-\infty}^\infty \frac{dx}{1+x^{2n}} = \frac{\pi}{n\sin(\pi/2n)} \quad (n = 1, 2, \ldots).$$

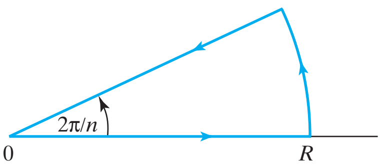


FIGURE 15.16 The contour for Exercise 12.

Integrands with poles on the real axis

In Exercises 13 through 22 evaluate the integrals using a contour comprising the semicircle $|z| = R$ in the upper half of the complex plane and a suitably indented real axis.

13. P.V. $\displaystyle\int_0^\infty \frac{\sin \pi x}{x(1-x^2)}\,dx.$

14. P.V. $\displaystyle\int_0^\infty \frac{\sin ax}{x(x^2+b^2)^2}\,dx \quad (b>0).$

15. P.V. $\displaystyle\int_0^\infty \frac{\cos ax - \cos bx}{x^2}\,dx \quad (a \ge 0, b \ge 0).$

16. P.V. $\displaystyle\int_{-\infty}^\infty \frac{\sin x}{(x^2+4)(x-1)}\,dx.$

17. P.V. $\displaystyle\int_0^\infty \frac{\sin ax}{x(x^2+b^2)}\,dx \quad (a, b>0).$

18. $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$ (Hint: Integrate the function $f(z) = [e^{2iz}-1]/z^2$).

19. $\displaystyle\int_0^\infty \frac{\sin^3 x}{x^3}\,dx$ (Hint: Integrate the function $f(z) = [e^{3iz}-3e^{iz}+2]/z^3$).

20. P.V. $\displaystyle\int_0^\infty \frac{x^2}{x^4-1}\,dx.$

21. P.V. $\displaystyle\int_0^\infty \frac{\cos ax}{1-x^4}\,dx \quad (a>0).$

22. P.V. $\displaystyle\int_0^\infty \frac{x}{x^4-1}\,dx.$

Integrands with branch points

In Exercises 23 through 28 evaluate the integrals by integrating around the contour in Fig. 15.14.

23. $\displaystyle\int_0^\infty \frac{x^\alpha}{x^2+1}\,dx \quad (-1<\alpha<1).$

24. $\displaystyle\int_0^\infty \frac{x^\alpha}{(x^2+1)^2}\,dx \quad (-1<\alpha<3, \alpha \neq 1).$

25. $\displaystyle\int_0^\infty \frac{x^{\alpha-1}}{1+x+x^2}\,dx \quad (0<\alpha<2).$

26. $\displaystyle\int_0^\infty \frac{dx}{x^\alpha(x+1)} \quad (0<\alpha<1).$

27. $\displaystyle\int_0^\infty \frac{x^{1/2}}{x^3+1}\,dx.$

28. $\displaystyle\int_0^\infty \frac{x^\alpha}{(x^2+1)^2}\,dx \quad (-1<\alpha<3).$

In Exercises 29 and 30 evaluate the integrals by integrating around the contour in Fig. 15.15.

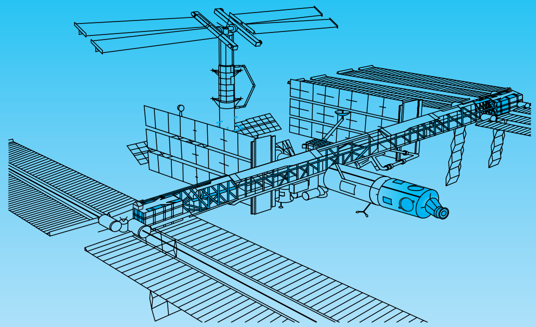
29. Show that

$$\int_0^\infty \frac{\ln x}{(1+x^2)^2}\,dx = -\frac{\pi}{4}.$$

30. Show that

$$\int_0^\infty \frac{(\ln x)^2}{1+x^2}\,dx = \frac{\pi^3}{8}.$$

CHAPTER 16



The Laplace Inversion Integral

When applying the Laplace transform to most practical problems, and obtaining the transform $F(s)$ of the required result, it is usually possible to find the required inverse transform $f(t)$ by using tables of Laplace transform pairs together with the operational properties listed in Chapter 7. Sometimes, however, the appropriate transform pairs cannot be found, so then some other way must be developed that enables the determination of the inverse Laplace transform. This is the problem that is addressed in the present chapter, where it is shown how the inversion of a Laplace transform can be performed by means of a special contour integral called the Laplace inversion integral.

The Laplace transform $F(s)$ of a function $f(t)$ is defined as

$$F(s) = \int_0^\infty e^{-st} f(t)dt,$$

provided $f(t)$ is such that the integral exists. The inversion of the Laplace transform to find the function $f(t)$ from a given transform $F(s)$ was performed in Chapter 7 by using a table of transform pairs together with the operational properties of the Laplace transform. In that approach the fact that in general the transform variable s is a complex variable was not used. However, when more complicated transforms $F(s)$ need to be inverted, and this cannot be achieved by using a table of transform pairs, it becomes necessary to regard $F(s)$ as a function of a complex variable and to use complex analysis to find $f(t)$.

This brief chapter uses complex analysis to derive an integral called the Laplace inversion integral that expresses $f(t)$ in terms of a contour integral involving $F(s)$. The inversion integral is then applied to some typical cases, where it is shown how the residues of the transform $F(s)$ can be used to recover the original function $f(t)$.

16.1 The Inversion Integral for the Laplace Transform

When the Laplace transform was introduced in Chapter 7, a table of Laplace transform pairs was developed by considering the transform variable s to be real, and these were then used with the operational properties of the Laplace transform to recover a wide variety of functions $f(t)$ from elementary Laplace

transforms $F(s)$. As tables of transform pairs do not always contain the required inverse Laplace transform and s must be allowed to be complex, some other method must be found by which to determine $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

The method we now derive shows that if $f(t)$ possesses a Laplace transform $F(s)$, so that

$$F(s) = \int_0^\infty e^{-st} f(t)dt, \tag{1}$$

where s can be complex, $f(t)$ can be recovered from its Laplace transform $F(s)$ by means of the complex line integral

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds, \tag{2}$$

the Laplace inversion integral

where $c > 0$ is a suitable real constant. The formula in (2) is called the **inversion integral** for the Laplace transform $F(s)$, and it involves an integral in the complex s-plane taken along the line $\mathrm{Re}\{s\} = c$ from minus infinity to infinity. We show later how this inversion integral can be evaluated in terms of the residues of $e^{st}F(s)$.

To establish result (2) we use the close relationship that exists between the complex form of the Fourier integral and the Laplace transform. The nature of this relationship can be seen from the fact that if

$$f(t) = \begin{cases} e^{-ct}g(t), & t > 0 \\ 0, & t < 0, \end{cases} \tag{3}$$

where the real constant $c > 0$ is chosen to guarantee the existence of $\mathcal{F}\{f(t)\}$, then from the definition of the complex form of the Fourier transform

$$\begin{aligned} \mathcal{F}\{f(t)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega t}f(t)dt = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-(c+i\omega)t}g(t)dt \\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-st}g(t)dt. \end{aligned} \tag{4}$$

The integral on the right of (4) is simply the Laplace transform of $g(t)$, though now the Laplace transform parameter $s = c + i\omega$ is complex. If F is the Fourier transform of f, the preceding result can be written

$$F(c + i\omega) = \frac{1}{\sqrt{2\pi}}\mathcal{L}\{g(t)\}. \tag{5}$$

derivation of the inversion integral

To derive the inversion integral (2) we start from the complex form of the Fourier integral representation for $f(t)$, which for clarity in the argument that follows we write as

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)e^{i\omega(t-u)}du\right]d\omega.$$

If we use the expression for $f(t)$ in (3), this becomes

$$e^{-ct}g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_0^\infty e^{i\omega t}e^{-(c+i\omega)u}g(u)du\right]d\omega.$$

However, as $e^{i\omega t}$ is not involved in the integral with respect to u, this can be rewritten as

$$e^{-ct}g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\left[\int_0^{\infty} e^{-su}g(u)du\right]d\omega,$$

where $s = c + i\omega$, showing that the integral in brackets is simply the Laplace transform $G(s)$ of $g(t)$ that exists by hypothesis. As $s = c + i\omega$, $ds = i d\omega$, so after the change of variable from ω to s in the integral with respect to ω, the limit $\omega = -\infty$ becomes $s = c - i\infty$, and the limit $\omega = \infty$ becomes $s = c + i\infty$, reducing the previous result to

$$e^{-ct}g(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{(s-c)t}G(s)ds.$$

Finally, cancelling the factor e^{-ct} that is not involved in the integral with respect to s, we arrive at the line integral

$$g(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}G(s)ds.$$

Apart from a change of notation, involving g and G in place of f and F, this is the inversion formula (2), so the derivation is complete. The function $g(t)$ will be independent of the value of c provided $\mathrm{Re}\{s\} > c$.

An important consequence of this derivation is that $g(t)$ can be allowed to be piecewise continuous with finite jump discontinuities. This follows because of the ability of the Fourier integral representation of a function to take account of finite jump discontinuities.

For the inversion integral to be useful, the line integral involved must be capable of evaluation in a straightforward manner, so let us now find how this can be accomplished. Consider the contour C_R in Fig. 16.1, where C_{1R} is the line $\mathrm{Re}\{s\} = c$, $-R \leq \mathrm{Im}\{s\} \leq R$, and C_{2R} is the semicircle $|s - c| = R$. If the integrand $e^{st}F(s)$ in (2) has a finite number of poles, all located inside C_R, then for sufficiently large R

$$\frac{1}{2\pi i}\int_{C_R} e^{st}F(s)ds = \Sigma\,\{\text{residues at each of the poles of } e^{st}F(s)\}.$$

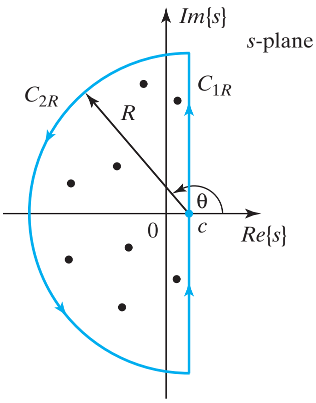


FIGURE 16.1 The contour C_R and a typical arrangement of poles inside C_R.

When expressed in terms of the contours C_{1R} and C_{2R} this result becomes

$$\frac{1}{2\pi i}\int_{c-iR}^{c+iR} e^{st}F(s)ds + \frac{1}{2\pi i}\int_{C_{2R}} e^{st}F(s)ds$$
$$= \Sigma\,\{\text{residues at each of the poles of } e^{st}F(s)\}.$$

Our objective will be to show that the integral around C_{2R} vanishes as $R \to \infty$. On C_{2R} we have $s = c + Re^{i\theta}$ with $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$, so after the change of variable $\theta = \frac{\pi}{2} + \phi$ we can write $s = c + iRe^{i\phi}$, $0 \le \phi \le \pi$, from which it follows that

$$ds = -Re^{i\phi}d\phi = -|s-c|e^{i\phi}d\phi.$$

Setting

$$I_R = \left|\frac{1}{2\pi i}\int_{C_{2R}} e^{st}F(s)ds\right|,$$

and transferring the modulus from outside the integral to inside, we arrive at the inequality

$$I_R \le \frac{1}{2\pi}\int_{C_{2R}} |e^{st}||F(s)||ds|$$
$$= \frac{1}{2\pi}\int_0^{\pi} |F(s)||\exp\{t[c + R(i\cos\phi - \sin\phi)]\}||s-c|d\phi.$$

Let us now suppose $F(s)$ is such that $|sF(s)| \le M$ on C_{2R} as $R \to \infty$. Then if we use the fact that for R sufficiently large $|s-c| \le |s| + |c| \le 2|s|$, the integral inequality becomes

$$I_R \le \frac{1}{\pi}\int_0^{\pi} |sF(s)|e^{ct}\exp(-Rt\sin\phi)d\phi = \frac{Me^{ct}}{\pi}\int_0^{\pi}\exp(-Rt\sin\phi)d\phi.$$

As $\sin\phi$ is symmetrical about the value $\frac{\pi}{2}$, this result can be rewritten as

$$I_R \le \frac{2Me^{ct}}{\pi}\int_0^{\pi/2}\exp(-Rt\sin\phi)d\phi.$$

Finally, applying the integral form of the Jordan inequality to this estimate, we find that

$$I_R \le \left(\frac{2Me^{ct}}{\pi}\right)\left(\frac{\pi}{2Rt}\right)(1 - e^{-Rt})$$

so, provided $t > 0$, this shows that $\lim_{R\to\infty} I_R = 0$. Consequently, in the limit as $R \to \infty$, we have shown that when $t > 0$,

$$\frac{1}{2\pi i}\int_{c-iR}^{c+iR} e^{st}F(s)ds = \Sigma\,\{\text{residue at each of the poles of } e^{st}F(s)\}.$$

This important result, which forms the next theorem, enables the inversion integral to be evaluated in terms of the residues of the function $e^{st}F(s)$.

THEOREM 16.1

Inversion of a Laplace transform by means of residues Let $F(s) = \mathcal{L}\{f(t)\}$, the Laplace transform of $f(t)$, be such that it has a finite number of poles, and choose c such that all the poles lie to the left of $\text{Re}\{s\} = c$. Then if a positive real number

the inversion integral and residues

M exists such that $|sF(s)| \leq M$ for all s to the left of $\text{Re}\{s\} = c$, the inverse Laplace transform $f(t) = \mathcal{L}^{-1}\{F(s)\}$ is given by

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \Sigma\,\{\text{residue at each of the poles of } e^{st}F(s)\}. \qquad \blacksquare$$

This theorem extends immediately to the case where $F(s)$ has an infinite number of poles all lying to the left of $\text{Re}\{s\} = c$ provided, as $R \to \infty$, the contour C_{2R} is allowed to expand in such a way that it never passes through a pole. The inversion of transforms of this type leads to the determination of $f(t) = \mathcal{L}^{-1}\{F(s)\}$ in the form of an infinite series of functions of t (see Example 16.4).

EXAMPLE 16.1

Use Theorem 16.1 to find $\mathcal{L}^{-1}\{(s^2 - a^2)/(s^2 + a^2)^2\}$, $a > 0$.

Solution Before applying Theorem 16.1 it is necessary to check that its conditions are satisfied. Using the contour in Fig. 16.1 and setting $F(s) = (s^2 - a^2)/(s^2 + a^2)^2$, the poles (double) of $F(s)$ are seen to be located at $s = \pm ia$, so for suitably large R they will lie inside the contour provided $\text{Re}\{s\} < c$, with $c > 0$. In addition, $\lim_{s\to\infty} |sF(s)| = 0$ when s lies to the left of the imaginary axis, so the conditions of Theorem 16.1 are satisfied.

Routine calculations show that the residues of $e^{st}F(s)$ at its two double poles are

$$\text{Res}\left\{\frac{e^{st}(s^2 - a^2)}{(s^2 + a^2)^2}, s = \pm ia\right\} = \frac{t}{2}\exp(\pm iat),$$

so

$$f(t) = \mathcal{L}^{-1}\left\{\frac{(s^2 - a^2)}{(s^2 + a^2)^2}\right\} = \frac{t}{2}\{\exp(iat) + \exp(-iat)\} = t\cos at,$$

confirming entry 12 in Table 7.1 of Laplace transform pairs. ■

If a Laplace transform involves a branch point, the contour in Fig. 16.1 must be modified by inserting a branch cut to make the function single valued inside the contour, and this often involves making a cut along the negative real axis. An inversion integral requiring a branch cut of this type is given in the next example.

EXAMPLE 16.2

Find $\mathcal{L}^{-1}\{1/\sqrt{s}\}$.

some typical examples

Solution The function $F(s) = 1/\sqrt{s}$ has a branch point at the origin of the s-plane, so instead of the contour in Fig. 16.1 we will use the contour in Fig. 16.2, where a branch cut has been made along the negative real axis with each side of the cut being connected by a small circular arc surrounding the branch point at the origin.

The semicircular contour C_{2R} in Fig. 16.1 is now replaced by the two circular arcs AB and EF of radius R together with the path BC along the top of the branch cut, the small circular arc Γ of radius ε around the branch point, and the path DE along the bottom of the branch cut. The function $F(s) = 1/\sqrt{s}$ is analytic and single valued inside this modified contour, which is bounded on the right by the vertical line C_{1R}. We will use the principal branch of the function for which the argument lies in the interval $-\pi < \theta \leq \pi$. As the branch cut along the negative real axis terminates at the origin, we must take $c > 0$.

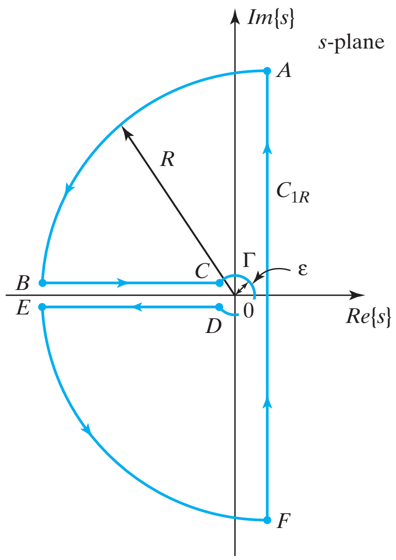


FIGURE 16.2 Modified contour with a branch cut to make $1/\sqrt{s}$ single valued.

On C_{2R} we have $s = c + Re^{i\theta}$ for $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$. For later use we now set $\theta = \pi/2 + \phi$, so s becomes $s = c + iRe^{i\phi}$, for $0 \le \phi \le \pi$. With this change of variable $ds = -Re^{i\phi}d\phi$, so $|ds| = Rd\phi$, and provided R is sufficiently large $|s| = |c + iRe^{i\phi}| \ge ||Re^{i\phi}| - |c|| = R - c$. We will also need to use the result that $|e^{st}| = |\exp\{t[(c - R\sin\phi) + iR\cos\phi]\}| = e^{ct}\exp\{-Rt\sin\phi\}$. The integral I_R around C_{2R} can now be estimated as follows:

$$I_R = \left| \int_{ABEF} \frac{e^{st}}{\sqrt{s}} ds \right| \le \int_{ABEF} \frac{|e^{st}|}{|s|^{1/2}} |ds| \le \frac{e^{ct} R}{(R-c)^{1/2}} \int_0^{\pi} \exp[-Rt\sin\phi] d\phi.$$

The symmetry of $\sin\phi$ about $\phi = \pi/2$ allows this to be rewritten as

$$I_R \le \frac{2e^{ct} R}{(R-c)^{1/2}} \int_0^{\pi/2} \exp[-Rt\sin\phi] d\phi,$$

so applying the integral form of the Jordan inequality we find that

$$I_R \le \frac{\pi e^{ct}}{(R-c)^{1/2} t}(1 - e^{-Rt}).$$

Allowing $R \to \infty$, with $t > 0$, in this last result shows that $\lim_{R\to\infty} I_R = 0$.

The integral around the contour Γ of radius ε, on which $s = \varepsilon e^{i\varphi}$, $ds = i\varepsilon e^{i\varphi} d\varphi$, and $s^{1/2} = e^{i\varphi/2}\sqrt{\varepsilon}$, is given by

$$\int_{-\pi}^{\pi} \frac{1}{e^{i\varphi/2}\sqrt{\varepsilon}} \exp[\varepsilon t(\cos\varphi + i\sin\varphi)] i\varepsilon e^{i\varphi} d\varphi,$$

but this also is seen to vanish as $\varepsilon \to 0$.

Along the top BC of the branch cut $s = re^{\pi i} = -r$, so $\sqrt{s} = e^{i\pi/2}\sqrt{r} = i\sqrt{r}$, and $ds = -dr$, whereas along the bottom DE of the cut $s = re^{-i\pi} = -r$, so $\sqrt{s} = e^{-i\pi/2}\sqrt{r} = -i\sqrt{r}$, and again $ds = -dr$. As no poles lie inside the contour, it follows

from the Cauchy integral theorem that

$$\frac{1}{2\pi i}\lim_{R\to\infty,\varepsilon\to 0}\left\{\int_{C_{1R}}\frac{e^{st}}{\sqrt{s}}ds+\int_{R}^{\varepsilon}\frac{1}{i\sqrt{r}}e^{-rt}(-dr)+\int_{\Gamma}\frac{e^{st}}{\sqrt{s}}ds\right.$$
$$\left.+\int_{\varepsilon}^{R}\frac{1}{(-i)\sqrt{r}}e^{-rt}(-dr)+\int_{C_{2R}}\frac{e^{st}}{\sqrt{s}}ds\right\}=0.$$

We have shown that when $t>0$ the third and last terms vanish in the limit as $R\to\infty$ and $\varepsilon\to 0$, so the equation reduces to

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{st}}{\sqrt{s}}ds=\frac{1}{2\pi i}\left\{-\int_{\infty}^{0}\frac{ie^{-rt}}{\sqrt{r}}dr+\int_{0}^{\infty}\frac{ie^{-rt}}{\sqrt{r}}dr\right\}=\frac{1}{\pi}\int_{0}^{\infty}\frac{e^{-rt}}{\sqrt{r}}dr.$$

The changes of variable $r=u^2$ followed by $v=u\sqrt{t}$ simplify this result to

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{st}}{\sqrt{s}}ds=\frac{2}{\pi\sqrt{t}}\int_{0}^{\infty}e^{-v^2}dv,$$

so using the standard result $\int_0^\infty e^{-v^2}dv=\sqrt{\pi}/2$ we find that

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\}=\frac{1}{\sqrt{\pi t}},\qquad \text{for } \operatorname{Re}\{s\}>0.$$

In the next example we consider a Laplace transform with an exponential factor in the numerator, which is known from the operational properties of the Laplace transform to arise from a shift in t.

EXAMPLE 16.3 Find $\mathcal{L}^{-1}\{e^{-s}/(s^2+1)\}$.

Solution It was shown in Chapter 7 that $\mathcal{L}^{-1}\{e^{-s}/(s^2+1)\}=H(t-1)\sin(t-1)$ for $t>0$, where $H(t-1)$ is the Heaviside unit step function defined as

$$H(t-a)=\begin{cases}0, & t<a\\ 1, & t>a.\end{cases}$$

how the inversion integral generates the Heaviside step function

We now show how the result $\mathcal{L}^{-1}\{e^{-s}/(s^2+1)\}$ can be recovered by means of the inversion integral. It is a routine matter to establish that Theorem 16.1 applies to the function $F(s)=e^{-s}/(s^2+1)$, which only has simple poles at $s=\pm i$, so we proceed directly to the determination of the residues of $e^{st}F(s)$. We have

$$\operatorname{Res}\left\{\frac{e^{s(t-1)}}{s^2+1},s=i\right\}=-\frac{i}{2}\exp[i(t-1)]$$

and

$$\operatorname{Res}\left\{\frac{e^{s(t-1)}}{s^2+1},s=-i\right\}=\frac{i}{2}\exp[-i(t-1)],$$

so from Theorem 16.1

$$f(t)=\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+1}\right\}=\left\{-\frac{i}{2}\exp[-i(t-1)]+\frac{i}{2}\exp[-i(t-1)]\right\}=\sin(t-1).$$

As the Laplace transform of a function $f(t)$ is not defined for $t<0$, we must require $\mathcal{L}^{-1}\{e^{-s}/(s^2+1)\}$ to be zero for $t<1$, so if we make use of the Heaviside

unit step function this becomes

$$f(t) = \mathcal{L}^{-1}\{e^{-s}/(s^2+1)\} = H(t-1)\sin(t-1) \quad \text{for } t > 0.$$

In this example a discontinuous function has been recovered from its Laplace transform by means of the inversion integral in (2). ■

The extension of Theorem 16.1 to a Laplace transform $F(s)$ with an infinite number of poles is illustrated in the following example.

EXAMPLE 16.4

Find $\mathcal{L}^{-1}\{\frac{1}{s\cosh s}\}$.

how the inversion integral generates a series

Solution Setting $F(s) = \frac{1}{s\cosh s}$, we see that $e^{st}F(s)$ has an infinite number of simple poles on the imaginary axis, with one at $s = 0$ due to the factor s in the denominator, and others at $s = (2n+1)\pi i/2$ with $n = 0, \pm 1, \pm 2, \dots$, corresponding to the zeros of $\cosh s$. As all the poles lie on the imaginary axis, when applying the inversion integral we will use the contour shown in Fig. 16.3 with $c > 0$ arbitrarily small, and to prevent the contour passing through a pole we set $R = k\pi$ with $k = 1, 2, \dots$.

Routine calculations show that

$$\text{Res}\{e^{st}F(s), s = 0\} = 1$$

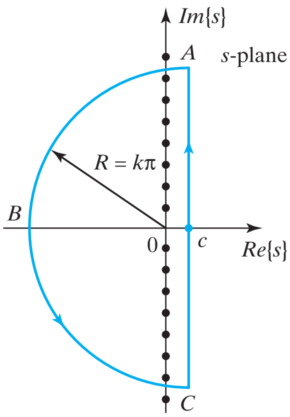


FIGURE 16.3 Contour containing poles on the imaginary axis.

and

$$\text{Res}\{e^{st}F(s), s = (2n+1)\pi i/2\} = (-1)^{n+1}\frac{2\exp[(2n+1)\pi i t/2]}{(2n+1)\pi}.$$

Extending Theorem 16.1 in an obvious manner we have

$$\begin{aligned} f(t) &= \frac{1}{2\pi i}\lim_{R\to\infty}\left\{\int_{-iR}^{iR}\frac{e^{st}}{s\cosh s}ds + \int_{ABC}\frac{e^{st}}{s\cosh s}ds\right\} \\ &= \Sigma\{\text{residues at poles of } e^{st}F(s)\}. \end{aligned}$$

On the semicircle ABC of radius R, $s = Re^{i\theta}$ with $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$, so $|s| = R$ and $|ds| = Rd\theta$. Substituting for s in e^{st} gives $|e^{st}| = \exp[Rt\cos\theta]$, and

$$\begin{aligned} |\cosh s| &= |\cosh(R\cos\theta)\cos(R\sin\theta) + i\sinh(R\cos\theta)\sin(R\sin\theta)| \\ &= [\cosh^2(R\cos\theta) - \sin^2(R\sin\theta)]^{1/2} \end{aligned}$$

The graph of $|\cosh s|$ as a function of θ is symmetrical about $\theta = \pi$ for all R, and it attains its least values at the ends of the interval $\pi/2 \le \theta \le 3\pi/2$. However, $R = k\pi$, so setting $\theta = \pi/2$ we find that on the semi-circle ABC

$$|\cosh s| \ge [1 - \sin^2(k\pi)]^{1/2} = 1.$$

Using these results to estimate the integral around ABC we find that

$$\begin{aligned} I_R &= \left|\int_{ABC}\frac{e^{st}}{s\cosh s}ds\right| \le \int_{ABC}\frac{|e^{st}|}{|s||\cosh s|}|ds| \le \int_{\pi/2}^{3\pi/2}\frac{\exp[k\pi t\cos\theta]}{k\pi}k\pi d\theta \\ &= \int_{\pi/2}^{3\pi/2}\exp[k\pi t\cos\theta]d\theta. \end{aligned}$$

After the change of variable $\theta = \pi/2 - \phi$ this becomes

$$I_R \le \int_0^{\pi} \exp[-k\pi t \sin\phi]d\phi,$$

but $\sin\phi$ is symmetric about $\phi = \pi/2$, so this is seen to be equivalent to

$$I_R \le 2\int_0^{\pi/2} \exp[-k\pi t \sin\phi]d\phi.$$

Applying the integral form of the Jordan inequality reduces this to

$$I_R \le \frac{1}{kt}(1 - e^{-k\pi t}),$$

so that provided $t > 0$, $\lim_{k\to\infty} I_R = 0$. Consequently we have shown that

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s\cosh s}\right\} = \sum \{\text{residues at the poles of } e^{st}F(s)\}.$$

Combining the residues of poles located at pairs of complex conjugate points along the imaginary axis causes the complex parts of the residues to cancel, leaving the real result

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s\cosh s}\right\} = 1 + \frac{4}{\pi}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{\cos[(2n+1)\pi t/2]}{2n+1}.$$

We see that in this case the inversion integral has given rise to a function $f(t)$ in the form of a sum of an infinite series of cosine functions.

To understand why this has occurred, we need only notice that $F(s)$ is, in fact, the Laplace transform of the rectangular pulse function

$$f(t) = 2\sum_{n=0}^{\infty}(-1)^n H[t - (2n+1)]$$

with period 4 and amplitude 2. So what has been recovered by the inversion integral is the Fourier series representation of the piecewise continuous function

$$f(t) = \begin{cases} 0, & 0 < t < 1 \\ 2, & 1 < t < 3 \\ 0, & 3 < t < 4, \end{cases}$$

where $f(t) = 0$ for $t < 0$ and $f(t+4) = f(t)$ for $t > 0$. ■

Although Theorem 16.1 provides a general formula for the inverse of a Laplace transform, it is not always easy to use. In certain cases the inversion integral can be avoided by employing a known transform together with one or more of the operational properties possessed by all Laplace transforms. This approach is illustrated in the next example.

EXAMPLE 16.5 Find $\mathcal{L}^{-1}\left\{\frac{1}{s\sqrt{s+1}}\right\}$.

Solution An attempt to find this inverse transform by means of Theorem 16.1 leads to difficulties in the determination of the residues, so we will employ a different approach. The first shift theorem for Laplace transforms asserts that if

$\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$, so by replacing s by $s+1$ in the result of Example 16.2 we have

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s+1}}\right\} = \frac{e^{-t}}{\sqrt{\pi t}}.$$

To complete the inversion process we now make use of the Laplace transform of an integral that asserts that if $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)d\tau.$$

Using this result with $\mathcal{L}^{-1}\{1/\sqrt{s+1}\}$ gives

$$\mathcal{L}^{-1}\left\{\frac{1}{s\sqrt{s+1}}\right\} = \frac{1}{\sqrt{\pi}}\int_0^t \frac{e^{-u}}{\sqrt{u}}du.$$

The change of variable $u = v^2$ converts this to

$$\mathcal{L}^{-1}\left\{\frac{1}{s\sqrt{s+1}}\right\} = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} \exp(-v^2)dv,$$

but the error function erf (x) is given by

$$\text{erf}\,(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-v^2)dv = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{n!(2n+1)},$$

so

$$\mathcal{L}^{-1}\left\{\frac{1}{s\sqrt{s+1}}\right\} = \text{erf}\,(\sqrt{t}).$$

EXAMPLE 16.6 Find

$$\mathcal{L}^{-1}\left\{\frac{\exp(-a\sqrt{s})}{s}\right\}, \quad \text{with } a > 0.$$

Solution The function has a branch point at the origin, so when evaluating the Laplace inversion integral by means of a contour integral it is necessary to use a contour with a cut along the negative real axis and to enclose the origin in a small circle of radius $\varepsilon > 0$. The complete contour C is shown in Fig. 16.4, and it comprises integrals along the path AB that in the limit will become the integral from $c - i\infty$ to $c + i\infty$, and the paths γ_1, Γ_1, C_1, Γ_2, and γ_2.

Setting

$$f(t) = \mathcal{L}^{-1}\left\{\frac{\exp(-a\sqrt{s})}{s}\right\} \quad \text{and} \quad F(s) = \frac{\exp(-a\sqrt{s})}{s}e^{st},$$

and noticing that $F(s)$ has no poles inside C, we can write

$$0 = \int_{AB} F(s)ds + \int_{\gamma_1} F(s)ds + \int_{\Gamma_1} F(s)ds + \int_{C_1} F(s)ds + \int_{\Gamma_2} F(s)ds + \int_{\gamma_2} F(s)ds,$$

and so

$$\frac{1}{2\pi i}\int_{AB} F(s)ds = \frac{1}{2\pi i}\left\{\int_{-\gamma_1} F(s)ds + \int_{-\Gamma_1} F(s)ds + \int_{-C_1} F(s)ds + \int_{-\Gamma_2} F(s)ds + \int_{-\gamma_2} F(s)ds\right\},$$

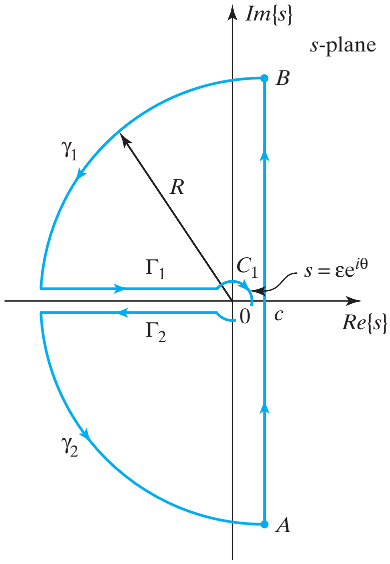


FIGURE 16.4 The contour involving a cut along the negative real axis.

where the symbols $-\gamma_1, -\Gamma, \ldots, -\gamma_2$ indicate the reversal of the direction of integration along these paths. In the limit as $A \to c - i\infty$ and $B \to c + i\infty$, the integral on the left becomes $f(t)$, and standard arguments show that as $R \to \infty$ the integrals along γ_1 and γ_2 that form part of the circle $|s| = R$ in Fig. 16.4 vanish. So, letting $R \to \infty$, the preceding result is seen to reduce to

$$\frac{1}{2\pi i}\int_{AB} F(s)ds = \frac{1}{2\pi i}\left\{\int_{-\Gamma_1} F(s)ds + \int_{-C_1} F(s)ds + \int_{-\Gamma_2} F(s)ds\right\}.$$

The path Γ_1 lies on the upper side of the negative real axis on which $s = re^{\pi i}$, so $\sqrt{s} = \sqrt{r}e^{i\pi/2} = i\sqrt{r}$. The path Γ_2 lies on the lower side of the negative real axis on which $s = re^{-\pi i}$, so $\sqrt{s} = \sqrt{r}e^{-i\pi/2} = -i\sqrt{r}$. Using these results and allowing for the reversal of the directions of integration, we have

$$f(t) = \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\left\{\int_{\varepsilon}^{\infty}\frac{\exp(-ia\sqrt{r})}{(-r)}e^{-rt}(-dr)\right.$$

$$+\int_{-\pi}^{\pi}\frac{\exp(-a\sqrt{\varepsilon}e^{i\theta/2})}{\varepsilon e^{i\theta}}\exp(\varepsilon te^{i\theta})\varepsilon ie^{i\theta}d\theta$$

$$\left.+\int_{\varepsilon}^{\infty}\frac{\exp(ia\sqrt{r})}{(-r)}e^{-rt}(-dr)\right\}.$$

Letting $\varepsilon \to 0$, the integral around the branch point becomes $\int_{-\pi}^{\pi} i\,d\theta = 2\pi i$, so after reversing the limits in the last integral the equation becomes

$$f(t) = \frac{1}{2\pi i}\left\{\int_0^{\infty}\frac{e^{-rt}}{r}(-2i)\sin\left(a\sqrt{r}\right)dr + 2\pi i\right\},$$

or

$$f(t) = 1 - \frac{1}{\pi}\int_0^{\infty}\frac{e^{-rt}}{r}\sin(a\sqrt{r})dr.$$

This expression can be put in a more convenient form if the integral

$$I = \frac{1}{\pi}\int_0^\infty \frac{e^{-rt}}{r}\sin\left(a\sqrt{r}\right)dr$$

is transformed by setting $rt = u^2$. After this change of variable the integral becomes

$$I = \frac{2}{\pi}\int_0^\infty \frac{\exp(-u^2)}{u}\sin(\beta u)du, \quad \text{where } \beta = a/\sqrt{t}.$$

Now

$$\frac{\partial I}{\partial \beta} = \frac{2}{\pi}\int_0^\infty \exp(-u^2)\cos(\beta u)du,$$

but from Exercise 24 in Exercise Section 14.3,

$$\int_0^\infty \exp(-u^2)\cos(\beta u)du = \frac{1}{2}\sqrt{\pi}\exp(-\beta^2/4),$$

so

$$\frac{\partial I}{\partial \beta} = \frac{1}{\sqrt{\pi}}\exp(-\beta^2/4).$$

Integration of this result from 0 to β, using the fact that $I = 0$ when $\beta = 0$, gives

$$I = \frac{1}{\sqrt{\pi}}\int_0^\beta \exp(-v^2/4)dv,$$

or

$$I = \frac{1}{\sqrt{\pi}}\int_0^{a/\sqrt{t}} \exp(-v^2/4)dv.$$

In terms of the error function

$$\operatorname{erf}(x) = \frac{2}{\pi}\int_0^x \exp(-t^2)dt,$$

integral I becomes

$$I = \operatorname{erf}\left(\frac{a}{2\sqrt{t}}\right),$$

and so

$$f(t) = 1 - \operatorname{erf}\left(\frac{a}{2\sqrt{t}}\right) = \operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right).$$

We have shown that

$$f(t) = \mathcal{L}^{-1}\left\{\frac{\exp(-a/\sqrt{s})}{s}\right\} = 1 - \operatorname{erf}\left(\frac{a}{2\sqrt{t}}\right) = \operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right). \quad \blacksquare$$

The inversion integral for the Laplace transform is discussed in some detail in reference [3.8] together with various applications, and also in references [4.3] and [4.4]. A comprehensive account of different forms of the Laplace transform and their associated inversion integrals is given in reference [3.18]; see also reference [6.10].

Summary

A contour integral called the Laplace inversion integral was derived that allows the function $f(t)$ to be recovered from its Laplace transform $F(s)$. This more advanced method

is necessary when the transform $F(s)$ is too complicated for $f(t)$ to be found by means of a table of transform pairs. The method was illustrated by being used to invert some more complicated transforms.

EXERCISES 16.1

In Exercises 1 through 13 use the inversion integral to find $\mathcal{L}^{-1}\{F(s)\}$.

1. $F(s) = \dfrac{1}{s(s^2+a^2)} \quad (a > 0).$

2. $F(s) = \dfrac{1}{(s+2)(s^2+4)}.$

3. $F(s) = \dfrac{(s-1)}{(s+1)^2}.$

4. $F(s) = \dfrac{4s+1}{s^2(s^2+1)}.$

5. $F(s) = \dfrac{1}{s^3(s+1)}.$

6. $F(s) = \dfrac{s}{(s+4)^2(s-1)}.$

7. $F(s) = \dfrac{1}{(s^2+a^2)^2} \quad (a > 0).$

8. $F(s) = \dfrac{1}{s^4-a^4} \quad (a > 0).$

9.* $F(s) = \dfrac{1}{s^{1/3}}$ (Hint: Use the gamma function in the final result).

10.* $F(s) = \dfrac{e^{-s}}{(s^2+1)^2}.$

11.* $F(s) = \dfrac{(s+1)e^{-2s}}{s^2-1}.$

12.* $F(s) = \dfrac{1}{\sqrt{s}(s-1)}.$

13.* $F(s) = \dfrac{1}{s\sqrt{s+a}} \quad (a > 0).$

14.* Find $\mathcal{L}^{-1}\{\frac{1}{s^{3/2}}\}$ without using the inversion integral by using a property of the Laplace transform that determines $\mathcal{L}^{-1}\{s^{-3/2}\}$ from the result $\mathcal{L}^{-1}\{s^{-1/2}\} = (\pi t)^{-1/2}$.

15.* Find $\mathcal{L}^{-1}\{\frac{1}{\sqrt{s+b}}\}$, and use the result with the convolution theorem for the Laplace transform to find $\mathcal{L}^{-1}\{\frac{1}{(s+a)\sqrt{s+b}}\} \quad (b > a > 0)$.

16.* Show that

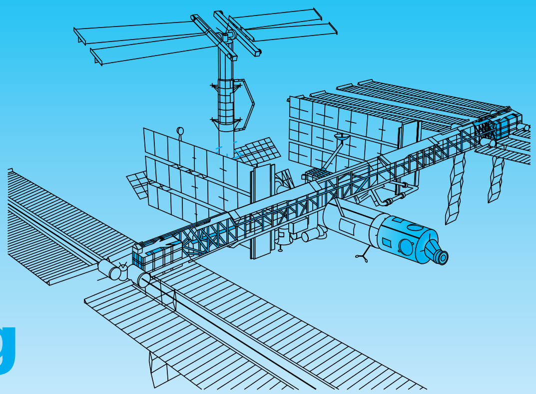
$$\mathcal{L}^{-1}\left\{\frac{1}{s^3 \sinh s}\right\} = \frac{t(t^2-1)}{6} - \frac{2}{\pi^3}\sum_{n=1}^{\infty}(-1)^n \frac{\sin n\pi t}{n^3}.$$

17.* Show that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(1+e^{-2as})}\right\} = \frac{\sin(t+a)}{2} + \frac{1}{a}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi t/2a}{1-(2n-1)^2\pi^2/4a^2} \quad (a > 0).$$

This Page Intentionally Left Blank

CHAPTER 17



Conformal Mapping and Applications to Boundary Value Problems

The way curves and regions in one plane are mapped by analytic functions onto another plane constitutes the study of conformal mappings. Conformal mappings concern the geometrical properties of analytic functions, and their study is closely related to the Laplace equation. This chapter defines a conformal mapping as one that preserves both the angle between intersecting curves and the sense of rotation from one curve to the other, and then proceeds to examine some of the most important examples of these mappings produced by elementary analytic functions.

Conformal mappings are shown to map a harmonic function in one plane into a harmonic function in another plane, and it is this property that is used when boundary value problems for the two-dimensional Laplace equation are solved. Applications of conformal mappings are made to two-dimensional boundary value problems involving heat flow, electrostatics, and ideal fluids.

Of particular interest is the ability of conformal mappings to map regions with a complicated boundary shape onto regions with a simple boundary shape. This is because such mappings can be used to solve two-dimensional boundary value problems for Laplace's equation in regions of complicated shape. The required solution follows directly from the fact that conformal mappings map one analytic function into another one. Consequently, if a conformal mapping can be found to map a complicated region onto one with a simple shape, once the solution of the corresponding boundary value problem in the simply shaped region has been found, it can be transformed back into the required solution in the complicated region.

17.1 Conformal Mapping

Let Γ_1 and Γ_2 be any two curves in the z-plane that radiate out from a common point of intersection P at z_0, as shown in Fig. 17.1a. Then if the curves have the respective parametric representations $z_1(t) = x_1(t) + iy_1(t)$ and $z_2(t) = x_2(t) + iy_2(t)$ for $a \le t \le b$, at their point of intersection P corresponding to $t = a$ we have $z_0 = z_1(a) = z_2(a)$. Now let the function $f(z)$ be a single-valued analytic function of z in some region D of the z-plane, and set $w = f(z)$. Then, as $f(z)$ is continuous, each point of Γ_1 will correspond to a unique point on some curve γ_1 in

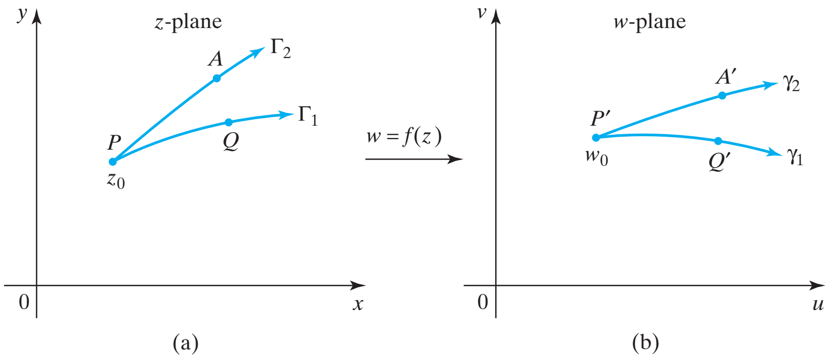


FIGURE 17.1 Mapping of curves Γ_1 and Γ_2 to γ_1 and γ_2 by $w = f(z)$.

the w-plane and, similarly, each point of Γ_2 will correspond to a unique point on some other curve γ_2 in the w-plane. As the curves Γ_1 and Γ_2 intersect at P located at z_0, the curves γ_1 and γ_2 must intersect at the point P', called the **image** of P, located at the point $w_0 = f(z_0)$ in the w-plane. In general, when points in the w-plane are identified by letters, their images in the w-plane are identified by using the same letters with the addition of a prime. So if A, B, C denote points in the z-plane, A', B', and C' will be used to denote the corresponding images in the w-plane.

As the parametrization in terms of t induces a **sense** (of direction) along the curves Γ_1 and Γ_2 as t increases, this sense is transferred to the curves γ_1 and γ_2 in the w-plane, as shown in Fig. 17.1b. Curves along which a sense of direction is defined are called **directed curves**.

image, directed curve, and conformal mapping

The curves Γ_1 and Γ_2 in the z-plane are said to be **mapped** onto the respective curves γ_1 and γ_2 in the w-plane by the function $w = f(z)$. It is usual to call γ_i the **image** of Γ_i under the mapping $w = f(z)$ from the z-plane to the w-plane and, conversely, as $f(z)$ is single valued, Γ_1 is called the image of γ_1 under the inverse mapping $z = f^{-1}(w)$ from the w-plane to the z-plane. In what follows we will show that for any z_0 such that $f'(z_0) \neq 0$, the analytic nature of $f(z)$ causes the mapping to preserve the angle of intersection between the curves Γ_1 and Γ_2 at P in the z-plane, so it equals the angle between their images γ_1 and γ_2 at P' in the w-plane. In addition, and equally important, we will show that the sense of rotation is preserved, so if the tangent to Γ_2 at P is obtained by rotating the tangent to Γ_1 at P counterclockwise through an angle α, then the tangent to γ_2 at P' is obtained by rotating the tangent to γ_1 at P' counterclockwise through the same angle α. A mapping that possesses these two properties is called a **conformal mapping**, and such mappings play a useful role in connection with the solution of boundary value problems for the two-dimensional Laplace equation.

To establish the conformal nature of the mapping produced by a single valued analytic function $w = f(z)$, we now appeal to Fig. 17.2. Consider the secant PQ on curve Γ_1 in Fig. 17.2a, and the corresponding secant $P'Q'$ in Fig. 17.2b, where Q is located at z_1 and Q' at $w_1 = f(z_1)$. In the limit as $Q \to P$, the angle between the secant PQ and the real axis in the z-plane becomes the angle α_1 between the tangent to Γ_1 at P and the real axis and, correspondingly, point $Q' \to P'$, causing the angle between the secant $P'Q'$ and the real axis in the w-plane to become the angle β_1 between the tangent to γ_1 at P' and the real axis.

Consequently, as $PQ = z_1 - z_0$, we can write

$$\alpha_1 = \lim_{z_1 \to z_0} \operatorname{Arg}(z_1 - z_0),$$

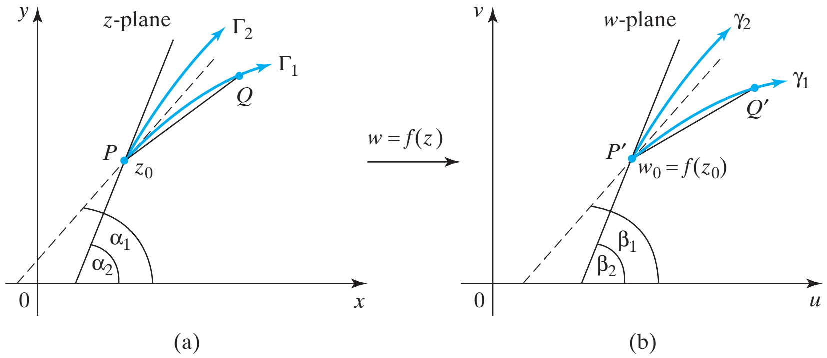


FIGURE 17.2 Secants PQ and $P'Q'$ in the z- and w-planes.

and correspondingly

$$\beta_1 = \lim_{z_1 \to z_0} \operatorname{Arg}(w_1 - w_0).$$

Forming the difference $\beta_1 - \alpha_1$, we have

$$\beta_1 - \alpha_1 = \lim_{z_1 \to z_0} \operatorname{Arg}(w_1 - w_0) - \lim_{z_1 \to z_0} \operatorname{Arg}(z_1 - z_0),$$

but $\operatorname{Arg} a - \operatorname{Arg} b = \operatorname{Arg}(a/b)$, so this last result can be written

$$\beta_1 - \alpha_1 = \lim_{z_1 \to z_0} \operatorname{Arg}\left(\frac{w_1 - w_0}{z_1 - z_0}\right).$$

As $f(z)$ is an analytic function, and so has a unique derivative $f'(z)$ irrespective of the way in which $z_1 \to z_0$, the preceding result shows that when $f'(z_0) \neq 0$,

$$\beta_1 - \alpha_1 = \operatorname{Arg} f'(z_0).$$

The uniqueness of the derivative $f'(z_0)$ means that the foregoing result is true for any other curve passing through P and its image curve through P', so, in particular, it is true for the curves Γ_2 and γ_2. We have shown

$$\beta_1 - \alpha_1 = \beta_2 - \alpha_2,$$

and this can be rewritten as

$$\alpha_2 - \alpha_1 = \beta_2 - \beta_1.$$

As the curves Γ_1 and Γ_2 were any two curves that intersect in the z-plane, this result has established the preservation of both the angles and their senses under the mapping $w = f(z)$, and hence the conformal nature of mappings produced by single-valued analytic functions at all points z where $f'(z) \neq 0$.

Although angles and senses of rotation are preserved by a conformal mapping, in general the length scale involved in a mapping at a point z_0 in the z-plane and at its image point $w_0 = f(z_0)$ in the w-plane is different. To find the linear scale factor $\rho(z_0)$ that is involved at $z = z_0$, we need to consider the limit of the quotient $|f(z) - f(z_0)|/|z - z_0|$ as $z \to z_0$, but this is simply $|f'(z_0)|$. So in a conformal mapping, provided $f'(z) \neq 0$, the **linear scale factor** $\rho(z)$ introduced at a point z when mapping infinitesimal line elements from the z-plane to the w-plane is $\rho(z) = |f'(z)|$ and, correspondingly, the **area scale factor** is $\rho^2(z)$. Because the scale factor and the

linear and area scale factors and critical points